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Abstract

We consider the Khokhlov-Zabolotskaya-Kuznetzov (KZK) equation $(u_t - uu_x - \beta u_{xx})_x - \gamma\Delta_y u = 0$, which describes for instance the propagation of sound beams in nonlinear media, in Sobolev spaces of functions periodic on x and with mean value zero. The derivation of KZK from the non linear isentropic Navier Stokes and Euler equations and approximation their solutions (in viscous and non viscous cases), the results of the existence, uniqueness, stability and blow-up of solution of KZK equation (using Alinhac's method) are obtained. We proved the existence of the shock wave for the problem with $\beta = 0$ (without viscosity and dissipation of energy). We have established the global existence in time of the beam's propagation in viscous media with $\beta > 0$ only for rather small initial data. In the proof of the existence of KZK solution the fractional step method have been analyzed. One justifies and gives as an example the numerical results of Thierry Le Pollès obtained by him in Laboratoire Ondes et Acoustique, ESPCI, Paris.

1 Introduction

The KZK equation, named after Khokhlov, Zabolotskaya and Kuznetsov, was originally derived as a tool for the description of nonlinear acoustic beams (cf for instance [10, 39]). It is used in acoustical problems as mathematical model that describes the pulse finite amplitude sound beam nonlinear propagation in the thermo-viscous medium, see for example [1, 24, 8, 9, 27]. Later it has been used in several other fields and in particular in the description of long waves in ferromagnetic media [33].

The KZK equation, as it have been demonstrated in [8], accurately describes the entire process of self-demodulation throughout the near field and into the far field, both on and off the axis of the beam (in water and glycerin). The

*Laboratoire Jacques-Louis Lions, Université Paris VI (rozanova@ann.jussieu.fr)

term “self-demodulation”, which was coined in the 1960s by Berktay, refers to the nonlinear generation of a low-frequency signal by a pulsed, high-frequency sound beam.

As it is known [9], the use of intense ultrasound in medical and industrial applications has increased considerably in recent years. Both plane and focused sources are used widely in either continuous wave or pules mode, and at intensities which lead to nonlinear effects such as harmonic generation and shock formation. Typical ultrasonic sources generate strong diffraction phenomena, which combine with finite amplitude effects to produce waveforms that vary from point to point within the sound beam. Nonlinear effects have become especially important at acoustic intensities employed in many current therapeutic and surgical procedures. In addition, biological media can introduce significant absorption of sound, which must also be considered. The KZK equation, as a nonlinear equation with effects of diffraction and of absorption, which can provide shock formation, is the mathematical model of these phenomena.

The non-linear phenomena found a recent application in the field of the ultrasonic medical imagery known under the name of “harmonic imagery”. In medical imagery where the echographic bars concentrates energy in a very narrow beam, the approach most commonly employed is the resolution of the equation KZK which describes focused beams.

In the present section the emphasis is put on the derivation of the equation for nonlinear acoustic in view of application to time reversal problems in nonlinear media. The KZK equation in its initial interpretation as in [10] is mostly studied by physicists but until now there are is no mathematical analysis of this problem. The KZK equation is not an integrable equation at variance Kadomtsev-Petviashvili (KP) equation known to be integrable. Numerically in [10] has been obtained the existence of a shock wave in the case of propagation of the beam in nondissipative media and a quasi shock wave for the dissipative media. The last phenomenon corresponds to the approximation of the beam’s front to the shock wave but the solution has the tentative to be global. We obtained the proof of the existence of the shock wave for the problem without viscosity. We have established the global existence in time of the propagation in viscous media only for rather small initial data. The announcement of the results can be found in [11, 12, 13].

This part is organized in the following way. First the derivation of the equation is borrowed from physical literature then the existence uniqueness stability of the equation is analyzed. Eventually a blow-up result which gives a limitation to the range of application is given as an adaptation of a result of [2], [3] and [4]. Using obtained results one proves a large time validity of the approximation for two cases: for non viscous thermoellastic media and viscous thermoellastic media.

Our main purpose is to prove existence and stability of solutions described by the KZK equation with the following properties

1. they are concentrated near the axis x_1;

2. they propagate along the x_1 direction;

3. they are generated either by initial condition or by a forcing on the boundary $x_1 = 0$.

This corresponds to the description of the quasi one d propagation of a signal in an homogenous but nonlinear isentropic media.

Therefore it is assumed that its variation in the direction

$$x' = (x_2, x_3, \ldots, x_n)$$

perpendicular to the x_1 axis is much larger that its variation along the axis x_1.

For instance for the linear wave equation in $\mathcal{R}^n$ $(n > 1)$:

$$\frac{1}{c^2}\partial_t^2 u - \Delta u = 0\,, \tag{1}$$

the following ansatz

$$u_\epsilon = U(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon} x') \tag{2}$$

involving a "profile"

$$U(\tau, z, y)$$

(with ϵ) small leads to the formula:

$$\partial^2_{\tau,z} U - \frac{1}{2}\Delta_y U = O(\epsilon), \tag{3}$$

or for functions $U(\tau, z, y) = A(z, y)e^{i\omega\tau}$, to the equation

$$i\omega\partial_z A - \frac{1}{2}\Delta_y A = O(\epsilon). \tag{4}$$

Observe that with $\epsilon = 0$ (3) and (4) are two variants of the classical paraxial approximation and that equation (3) contains the linear non diffusive terms of the KZK equation which usually has the following form for some positive constants β and γ:

$$\partial^2_{\tau,z} U - \frac{1}{2}\partial_\tau^2 U^2 - \beta\partial_\tau^3 U - \gamma\Delta_y U = 0.$$

On the other hand the isentropic evolution of a thermo-elastic non viscous media is given by the following Euler Equation:

$$\partial_t\rho + \nabla(\rho v) = 0\,, \quad \rho(\partial_t v + v\cdot\nabla v) = -\nabla p(\rho)\,. \tag{5}$$

Any constant state (ρ_0, v_0) is a stationary solution of (5). Linearization near this state introduces the variables

$$\rho = \rho_0 + \epsilon\tilde{\rho}\,, \quad v = v_0 + \epsilon\tilde{v}$$

and the acoustic system:

$$\partial_t\tilde{\rho} + \rho_0\nabla\tilde{v} = 0\,, \quad \rho_0\partial_t\tilde{v} + p'(\rho_0)\nabla\tilde{\rho} = 0, \tag{6}$$

which is equivalent to the wave equation:

$$\frac{1}{c^2}\partial_t^2\tilde{\rho} - \Delta\tilde{\rho} = 0\,, \quad \partial_t\tilde{v} = -\frac{p'(\rho_0)}{\rho_0}\nabla\tilde{\rho}, \tag{7}$$

where $c = \sqrt{p'(\rho_0)}$ is the sound speed of the unperturbed media.

And observe that the equation (3) which is the linearized and non viscous part of the KZK equation can be obtained in two steps. First consider small perturbations of a constant state for the isentropic Euler equation which are solution of the acoustic equation and then consider a paraxial approximation of such solutions.

The derivation of the full KZK equation follows almost the same line. It takes into account the viscosity and the size of the nonlinear terms. One starts from a Navier Stokes system:

$$\partial_t\rho + \nabla(\rho u) = 0\,, \quad \rho[\partial_t u + (u\cdot\nabla)\,u] = -\nabla p(\rho,S) + b\Delta u\,, \tag{8}$$

the pressure is given by the state law $p = p(\rho, S)$, where S is entropy.

First one assumes that the temperature T and the entropy S have the small increments $\tilde{T}$ and $\tilde{S}$. With the hypothesis of potential motion one introduces constant states

$$\rho = \rho_0, \quad u = u_0.$$

Next one assumes that the fluctuation of density (around the constant state ρ_0), of velocity (around u_0, which can be taken equal to zero with galilean), are of the same order ϵ:

$$\rho_\epsilon = \rho_0 + \epsilon\tilde{\rho}_\epsilon\,, \quad u_\epsilon = \epsilon\tilde{u}_\epsilon\,, \quad b = \epsilon\tilde{b},$$

here ϵ is a dimensionless parameter which characterizes the smallness of the perturbation. For instance in water with a initial power of the order of $0.3\,\mathrm{Vt/cm}^2$ $\epsilon = 10^{-5}$. Using the transport heat equation in the form

$$\rho_0 T_0\frac{\partial\tilde{S}}{\partial t} = \kappa\triangle\tilde{T},$$

the approximate state equation

$$p = c^2\epsilon\tilde{\rho}_\epsilon + \frac{1}{2}\left(\frac{\partial^2 p}{\partial\rho^2}\right)_S\epsilon^2\tilde{\rho}_\epsilon^2 + \left(\frac{\partial p}{\partial S}\right)_\rho\tilde{S}$$

(where the notation $(\cdot)_S$ means that the expression in brackets is constant on S), can be replaced [10], thanks to the relation

$$\tilde{S} = -\frac{\kappa}{T_0}\left(\frac{\partial T}{\partial p}\right)_S \operatorname{div} u_\epsilon,$$

by

$$p = c^2\epsilon\tilde{\rho}_\epsilon + \frac{(\gamma-1)c^2}{2\rho_0}\epsilon^2\tilde{\rho}_\epsilon^2 - \kappa\left(\frac{1}{C_v} - \frac{1}{C_p}\right)\nabla.u_\epsilon. \tag{9}$$

The system (8) becomes an isentropic system

$$\partial_t \rho_\epsilon + \nabla(\rho_\epsilon u_\epsilon) = 0\,, \quad \rho[\partial_t u_\epsilon + (u_\epsilon \cdot \nabla)\, u_\epsilon] = -\nabla p(\rho_\epsilon) + \epsilon\nu\Delta u_\epsilon\,, \tag{10}$$

with the approximate state equation

$$p = p(\rho_\epsilon) = c^2 \epsilon \tilde{\rho}_\epsilon + \frac{(\gamma-1)c^2}{2\rho_0}\epsilon^2 \tilde{\rho}_\epsilon^2 \tag{11}$$

and a rather small and positive viscosity coefficient:

$$\epsilon\nu = b + \kappa\left(\frac{1}{C_v} - \frac{1}{C_p}\right).$$

Next one reminds the direction of propagation of the beam say along the axis x_1, and therefore considers the following profiles:

$$\tilde{\rho}_\epsilon = I(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x')\,, \tag{12}$$

$$\tilde{u}_\epsilon = (u_{\epsilon,1}, u'_\epsilon) = (v(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x'), \sqrt{\epsilon}\vec{w}(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x'))\,. \tag{13}$$

In (12) and (13) the argument of the functions will be denoted by $(\tau\,, z\,, y)$ and c is taken equal to the sound speed $c = \sqrt{p'(\rho_0)}$. Inserting the functions $\rho_\epsilon = \rho_0 + \epsilon I$, u_ϵ in the system (10) one obtains:

1 For the conservation of mass:

$$\begin{aligned}&\partial_t \rho_\epsilon + \nabla(\rho_\epsilon u_\epsilon) = \epsilon(\partial_\tau I - \frac{\rho_0}{c}\partial_\tau v) + \\ &+\epsilon^2\left(\rho_0(\partial_z v + \nabla_y \cdot \vec{w}) - \frac{1}{c}v\partial_\tau I - \frac{1}{c}I\partial_\tau v\right) + O(\epsilon^3) = 0\,.\end{aligned} \tag{14}$$

2 For the conservation of momentum in the x_1 direction:

$$\begin{aligned}&\rho_\epsilon\epsilon(\partial_t u_{\epsilon,1} + u_\epsilon \nabla u_{\epsilon,1}) + \partial_{x_1}p(\rho_\epsilon) - \epsilon^2\nu\Delta u_{\epsilon,1} = \epsilon(\rho_0\partial_\tau v - c\partial_\tau I) + \\ &+\epsilon^2\left(I\partial_\tau v - \tfrac{\rho_0}{c}v\partial_\tau v + c^2\partial_z I - \tfrac{(\gamma-1)}{2\rho_0}c\partial_\tau I^2 - \tfrac{\nu}{c^2}\partial_\tau^2 v\right) + O(\epsilon^3) = 0.\end{aligned} \tag{15}$$

And finally for the orthogonal (to the axis x_1) component of the momentum one has:

$$\begin{aligned}&\rho_\epsilon\epsilon(\partial_t u'_\epsilon + u_\epsilon\nabla u'_\epsilon) + \partial_{x'}p(\rho_\epsilon) - \epsilon^2\nu\Delta u'_\epsilon = \epsilon^{\frac{3}{2}}(\rho_0\partial_\tau\vec{w} + c^2\nabla_y I) + \\ &+\epsilon^{\frac{5}{2}}(-\tfrac{\rho_0 v}{c}\partial_\tau\vec{w} + I\partial_\tau\vec{w} + \tfrac{(\gamma-1)c^2}{2\rho_0}\nabla_y I^2 - \tfrac{\nu}{c^2}\Delta_y\vec{w}) + O(\epsilon^3) = 0.\end{aligned} \tag{16}$$

To eliminate the terms of the first order in ϵ we need to pose:

$$\partial_\tau I - \frac{\rho_0}{c}\partial_\tau v = 0, \tag{17}$$

which also implies

$$\rho_0\partial_\tau v - c\partial_\tau I = 0,$$

and therefore I and v should be related by the formula:

$$v = \frac{c}{\rho_0} I \tag{18}$$

and the second order terms of (14) and (15) by the formula:

$$\begin{aligned}\rho_0(\partial_z v + \nabla_y\cdot\vec{w}) - \tfrac{1}{c}v\partial_\tau I - \tfrac{1}{c}I\partial_\tau v =\\ = -\tfrac{1}{c}(I\partial_\tau v - \tfrac{\rho_0}{c}v\partial_\tau v + c^2\partial_z I - \tfrac{(\gamma-1)}{2\rho_0}c\partial_\tau I^2 - \tfrac{\nu}{c^2}\partial_\tau^2 v),\end{aligned} \tag{19}$$

which (with (18)) gives:

$$\rho_0\nabla_y\cdot\vec{w} + 2c\partial_z I - \frac{(\gamma+1)}{2\rho_0}\partial_\tau I^2 - \frac{\nu}{c^2\rho_0}\partial_\tau^2 I = 0. \tag{20}$$

Eventually one uses the equation of the orthogonal moment (16) to eliminate the term $\rho_0\nabla_y\cdot\vec{w}$. Assume in agreement with (16) that

$$\rho_0\partial_\tau\vec{w} + c^2\nabla_y I = 0, \tag{21}$$

take the divergence with respect to y of this equation. Differentiate (20) with respect to τ, and combine to obtain:

$$c\partial_{\tau z}^2 I - \frac{(\gamma+1)}{4\rho_0}\partial_\tau^2 I^2 - \frac{\nu}{2c^2\rho_0}\partial_\tau^3 I - \frac{c^2}{2}\Delta_y I = 0. \tag{22}$$

The KZK equation (22) is written for the perturbation of density, but the same equation with only different constants can be also derived for the pressure and the velocity. The passage between these KZK equations is possible thanks to (11), (18) and (21). For example the equation for the pressure has the form

$$\partial_{\tau z}^2 p - \frac{\beta}{2\rho_0 c^3}\partial_\tau^2 p^2 - \frac{\delta}{2c^3}\partial_\tau^3 p - \frac{c}{2}\triangle_y p = 0.$$

The above derivation is standard in physic articles however it does not imply that the function

$$\rho_\epsilon = \rho_0 + \epsilon I, u_\epsilon = \epsilon(v, \sqrt{\epsilon}\vec{w})$$

is a solution of the system (10) with an error term of the order of ϵ^3. In fact one can assume (17) and that (21) with (20) take place, but not the fact that this quantity which corresponds to the term of the order of ϵ^2 both in the conservation of mass and momentum along the axis x_1 is zero. To remedy to this fact and also to ensure an error of the order of $\epsilon^{\frac{5}{2}}$ in the moment orthogonal to the x_1 direction one introduces an Hilbert expansion type construction and writes

$$\rho_\epsilon = \rho_0 + \epsilon I\,, \quad u_\epsilon = \epsilon(v + \epsilon v_1, \sqrt{\epsilon}\vec{w}), \tag{23}$$

assuming that I is solution of the KZK equation (22), while v and w are given in term of I by (18) and (21), one obtains, modulo terms of order $\epsilon^{\frac{5}{2}}$, for the right hand side of the equations (14), (15) and (16):

$$\epsilon^2\left(-\frac{\rho_0}{c}\partial_\tau v_1+\rho_0(\partial_z v+\nabla_y\cdot w)-\frac{1}{\rho_0}\partial_\tau I^2\right),$$
$$\epsilon^2\left(\rho_0\partial_\tau v_1+c^2\partial_z I-\frac{\gamma-1}{2\rho_0}c\partial_\tau I^2-\frac{\nu}{c\rho_0}\partial_\tau^2 I\right).$$

Taking into account the KZK equation this implies for the "corrector" v_1 the relation:

$$\partial_\tau v_1=\frac{\gamma-1}{2\rho_0^2}c\partial_\tau I^2+\frac{\nu}{c\rho_0^2}\partial_\tau^2 I-\frac{c^2}{\rho_0}\partial_z I. \tag{24}$$

At this point one can state a theorem with hypothesis to be specified later in section 3 (see theorems 7,8, 10).

Theorem 1 *Let I be a smooth solution of the KZK equation (22), define the functions v, w and v_1 by the known I. Define the function $\overline{U}_\epsilon=(\overline{\rho}_\epsilon,\overline{u}_\epsilon)$ by the formula:*

$$(\overline{\rho}_\epsilon,\overline{u}_\epsilon)(x_1,x',t)=(\rho_0+\epsilon I,\epsilon(v+\epsilon v_1,\sqrt{\epsilon}\vec{w}))(t-\frac{x_1}{c},\epsilon x_1,\sqrt{\epsilon}x').$$

Then there exist constants $C\geq 0$ and $T_0=O(1)$, such that for any finite time $0<t<T_0\frac{1}{\epsilon}\ln\frac{1}{\epsilon}$ and $\epsilon>0$, there exists a smooth solution $U_\epsilon=(R_\epsilon,U_\epsilon)(x,t)$ of the isentropic Navier-Stokes equation such that one has for some $s\geq 0$:

$$\|\overline{U}_\epsilon-U_\epsilon\|_{H^s}\leq \epsilon^{\frac{5}{2}}e^{\epsilon Ct}.$$

It is interesting to notice that for the non viscous case, i.e., for the isentropic compressible Euler system, the KZK like equation with $\beta=0$ have been obtained using the scaling of nonlinear diffractive geometric optic theory in [16, p. 1233] (in 2d) in the framework of nonlinear diffractive geometric optic with rectification. The initial goal of the article is to construct the nonlinear symmetric hyperbolic equation

$$L(u,\partial_x)u+F(u)=0,$$

and the case of the isentropic compressible Euler system is given as an example. The basic ansatz in [16] has three scales

$$u_\epsilon(x)=\epsilon^2 a\left(\epsilon,\epsilon x,x,\frac{x\cdot\beta}{\epsilon}\right),$$

where

$$a(\epsilon,X,x,\theta)=a_0(X,x,\theta)+\epsilon a_1(X,x,\theta)+\epsilon^2 a_2(X,x,\theta).$$

Here $x=(t,y)\in\mathcal{R}^{1+d}$, $\beta=(\tau,\eta)\in\mathcal{R}^{1+d}$ and the profiles $a_j(X,x,\theta)$ are periodic in θ. The KZK like equation of the form

$$\partial_T a-\triangle_y\partial_\theta^{-1}a+\sigma a\partial_\theta a=0,$$

with $\sigma\in\mathcal{R}$ determined from some identity and with $T=\frac{t}{\epsilon}$, holds for the profile a_0 with mean value zero on θ (for the proof see [16, pp. 1231, 1234]) which corresponds to vanishing non oscillatory part, if we have in our mind the notation of [16, p.1181]:

if (see [16, p.1181]) $\underline{a}:=\frac{1}{2\pi}\int_0^{2\pi}a d\theta$, the oscillating part is denoted $a^*:=a-\underline{a}$.

The analogue technique is used in [38] to study the short wave approximation for general symetric hyperbolic systems as

$$\begin{cases} L(\partial)u=\tilde{F}(u)\partial_x u, & \text{with } (x,y)\in\mathcal{R}\times\mathcal{R},\\ u(0)=\epsilon u^0(x/\epsilon,y)\in\mathcal{R}^n. \end{cases}\tag{25}$$

with an hyperbolic operator $L(\partial)=\partial_t+A\partial_x+B\partial_y+E$. Short waves stands for short-wavelenght approximate solutions, or equivalently approximate solutions with initial data whose oscillatory frequencies are large compared to the paremeters of the system. For the variables

$$T=\frac{t}{\epsilon},\ X=\frac{x}{\epsilon},\ y,\tau=\epsilon t\tag{26}$$

in [38] one looks for the approximate solutions in the form

$$u^\epsilon(t,x,y)=\epsilon(u_0+\epsilon u_1+\epsilon^2u_2)(T,X,y,\tau).$$

For the first profile u_0 one has the system of the form

$$\begin{cases} (\partial_T+c\partial_X)u_0=0,\\ (\partial_\tau\partial_X-\partial_Y^2)u_0=\partial_X(u_0\partial_X u_0). \end{cases}\tag{27}$$

corresponding to the KZK equation for the function $\tilde{u}_0(X-cT,\tau,y,X)$. The estimate of the approximate result in [38] between the exact solution v^ϵ of (25) and the solution u_0^ϵ of a system of the form (27) is following

$$\frac{1}{\epsilon}\|v^\epsilon-\epsilon u_0^\epsilon\|_{L_\infty([0,\tau_0/\epsilon]\times\mathcal{R}^2_{x,y})}=o(1).$$

To analyze common points between this work and KZK-approximation we can pass from the variables corresponding to our scaling

$$(t-\frac{x_1}{c},\epsilon x_1,\sqrt{\epsilon}x')$$

to "$\widetilde{\text{variable}}=\sqrt{\epsilon}\,\text{variable}$" in following way

$$\left(\frac{1}{\sqrt{\epsilon}}(\tilde{t}-\frac{\tilde{x}_1}{c}),\sqrt{\epsilon}\tilde{x}_1,\tilde{x}'\right)$$

and supposing now that $\epsilon = \tilde{\epsilon}^2$, i.e., we obtain

$$\left(\frac{1}{\tilde{\epsilon}}(\tilde{t} - \frac{\tilde{x}_1}{c}), \tilde{\epsilon}\tilde{x}_1, \tilde{x}'\right),$$

and similar

$$(\rho, u) = (\rho_0 + \tilde{\epsilon}\tilde{I}, \tilde{\epsilon}(\tilde{v} + \tilde{\epsilon}^2\tilde{v}_1, \tilde{\epsilon}\tilde{w})).$$

This variables exactly correspond to Texier's case [38]

$$(T - \frac{X}{c}, \tau, y).$$

To the first profile ϵu_0 from [38] there corresponds to $(\rho_0 + \tilde{\epsilon}\tilde{I}, \tilde{\epsilon}\tilde{v})$ for which we have exactly the system (27) in the form of (17) and the KZK equation without viscous therm. The profile $\tilde{\epsilon}^2\tilde{w}$ is associated to $\epsilon^2 u_1$ and the profile $\tilde{\epsilon}^3\tilde{v}_1$ is associated to $\epsilon^3 u_2$. The result of [38] is obtained for nonperiodic case and without the vanishing mean condition important for physical reasons.

This small analysis of the abstract works shows that our approach is similar where the variables have been switched with ϵ "$\widetilde{\text{variable}} = \sqrt{\epsilon}\,\text{variable}$" to balance the oscillation :

$$\frac{1}{\sqrt{\epsilon}}\left(\tilde{t} - \frac{\tilde{x}_1}{c}\right) \mapsto \left(t - \frac{x_1}{c}\right).$$

In other words we can say that we have $O(1)$ oscillations.

The scaling of Sanchez [33] for Landau-Lifshitz-Maxwell equations in $\mathcal{R}^3$ is very different. Sanchez starts by the system

$$\partial_t M = -M \wedge H - \frac{\gamma}{|M|} M \wedge (M \wedge H) \ \text{ in } \mathcal{R}^3, \tag{28}$$

$$\partial_t (H + M) - \nabla \wedge E = 0 \ \text{ in } \mathcal{R}^3, \tag{29}$$

$$\partial_t E + \nabla \wedge H = 0 \ \text{ in } \mathcal{R}^3, \tag{30}$$

which represents the Landau-Lifshitz-Maxwell equations and admits stable equilibrium solutions where the magnetization is uniform and everywhere parallel to the effective magnetic field:

$$(M, H, E)_\alpha = (M_0, \alpha^{-1} M_0, 0), \ \alpha > 0. \tag{31}$$

Then he is interesting in small perturbations of the equilibrium states whose size is measured by a small parameter ϵ. The perturbation is taken in the form:

$$\begin{aligned} M(t, x, y) &= M_0 + \epsilon^2 \widetilde{M}(\tilde{t}, \tau, \tilde{x}, \tilde{y}), \\ H(t, x, y) &= \alpha^{-1} M_0 + \epsilon^2 \widetilde{H}(\tilde{t}, \tau, \tilde{x}, \tilde{y}), \\ E(t, x, y) &= \epsilon^2 \widetilde{E}(\tilde{t}, \tau, \tilde{x}, \tilde{y}), \end{aligned}$$

where $\tilde{t}$, τ, $\tilde{x}$, $\tilde{y}$ are new rescaled variables:

$$\tilde{t} = \epsilon^2 t, \ \tau = \epsilon^4 t, \ \tilde{x} = \epsilon^2 x, \ \tilde{y} = \epsilon^3 y. \tag{32}$$

We notice here that if we take

$$\hat{t} = \epsilon^3 t,\ \hat{x} = \epsilon^3 x,\ \hat{y} = \epsilon^3 y,$$

we exactly obtain (26) from (32):

$$\tilde{t} = \frac{\hat{t}}{\epsilon},\ \tau = \epsilon\hat{t}\ \tilde{x} = \frac{\hat{x}}{\epsilon},\ \tilde{y} = \hat{y}.$$

But the smallness of the functions' profiles are different. The vector of perturbation $U(\tilde{t},\tau,\tilde{x},\tilde{y}) = \left(\alpha^{-\frac{1}{2}}\widetilde{M},\widetilde{H},\widetilde{E}\right)^t(\tilde{t},\tau,\tilde{x},\tilde{y})$ satisfy an equation in the form ($\sim$ on the variables is omitted)

$$\begin{aligned}&\partial_t U + \epsilon^2\partial_\tau U + A_1\partial_x U + \epsilon A_2\partial_y U + \epsilon^{-2}LU = B(U,U) + \epsilon^2 T(U,U,U),\\ &U(0,0) = U_0^0,\end{aligned}$$

where A_1, A_2, L are linear operators in $\mathcal{R}^9$, B is a bilinear map on $\mathcal{R}^9\times\mathcal{R}^9$, and T is a trilinear map on $\mathcal{R}^9\times\mathcal{R}^9\times\mathcal{R}^9$. Using then an asymptotic formal expansion of U in the form $U \equiv U_0 + \epsilon U_1 + \epsilon^2 U_2 + ...$, Sanchez obtains that the leading term U_0 breaks down in five traveling and standing terms, $U_0 = \sum_{j=1}^5 u_j$, which satisfy the transport equation and the Zabolotskaya-Khokhlov equation:

$$\begin{aligned}&(\partial_t + v_j\partial_x)u_j = 0,\\ &\partial_x(\partial_\tau u_j - D_j\partial_x^2 u_j + B_j(u_j,\partial_x u_j) + F_j(u_j,u_j,u_j)) = C_j\partial_y^2 u_j\end{aligned}$$

with constant v_j which is the speed of the wave in the direction k (the x-direction). He also proves the validation of this approximation for the time of order $\frac{T_0}{\epsilon^2}$.

Remark 1 *Several limits of the equation (37) leads to classical PDE.*
- *With $\rho_0 c \to \infty$ it becomes the paraxial approximation:*

$$\partial^2_{\tau z} I - \frac{c}{2}\Delta_y I = 0, \tag{33}$$

or in term of the pressure the equation

$$\frac{\partial p}{\partial z} = \frac{c}{2}\int_0^\tau \triangle_y p\, d\tau'.$$

The solutions of these equations have been numerically computed by Thierry Le Pollès (in Laboratoire Ondes et Acoustique, ESPCI, Paris) using a fractional step method. The proof of the validity of this method will be given in section 2.1.2. The figures 2 and 3 have been simulated for the three dimensional problem for pressure p of a sound beam propagating in the water

$$\frac{\partial p}{\partial z} = \frac{c}{2}\int_0^\tau \triangle_y p\, d\tau', \quad y = (y_1, y_2),$$

$$p(\tau, 0, y) = g(\tau) \quad y \in \Omega, \quad \tau > 0,$$
$$\frac{\partial p}{\partial n} = 0 \text{ for } \partial\Omega, \quad \tau > 0.$$

Here $g(\tau)$ is the signal of the source situated in $z = 0$ and

$$g(\tau) = P_0 \exp[-(2\tau/T_d)^{2m}] \sin(w_0 t). \tag{34}$$

- *And when I does not depend on y it is the Burgers-Hopf equation*

$$c\partial_z I - \frac{(\gamma+1)}{4\rho_0}\partial_\tau I^2 - \frac{\nu}{2c^2\rho_0}\partial_\tau^2 I = 0 \tag{35}$$

and eventually in this case with $\nu = 0$ the Burgers equation:

$$c\partial_z I - \frac{(\gamma+1)}{4\rho_0}\partial_\tau I^2 = 0.$$

In term of the pressure fluctuation, (35) is

$$\frac{\partial p}{\partial z} = \frac{\delta}{2c^3}\frac{\partial^2 p}{\partial \tau^2} + \frac{\beta}{2\rho_0 c^3}\frac{\partial p^2}{\partial \tau^2}. \tag{36}$$

The numerical simulation of the solution of (36) with the same initial and boundary data as in (33) is given in the figures 4 and 5.

- *The analogous* 2d *version of KZK equation is*

$$c\partial_{\tau z}^2 I - \frac{(\gamma+1)}{4\rho_0}\partial_\tau^2 I^2 - \frac{\nu}{2c^2\rho_0}\partial_\tau^3 I - \frac{c^2}{2}\partial_y^2 I = 0.$$

And for a "beam" (rotationally invariant around the x_1 axis) in 3 *space variables it is:*

$$c\partial_{\tau z}^2 I - \frac{(\gamma+1)}{4\rho_0}\partial_\tau^2 I^2 - \frac{\nu}{2c^2\rho_0}\partial_\tau^3 I - \frac{c^2}{2}(\partial_r^2 I + \frac{1}{r}\partial_r I) = 0. \tag{37}$$

The figures 6 and 7 represent the graph of the solution of the full KZK equation, composed of the both parts of (33) and (36) with the source (34)

$$\frac{\partial p}{\partial z} = \frac{c}{2}\int_0^\tau \triangle_y p d\tau' + \frac{\delta}{2c^3}\frac{\partial^2 p}{\partial \tau^2} + \frac{\beta}{2\rho_0 c^3}\frac{\partial p^2}{\partial \tau^2}. \tag{38}$$

All figures 2-7 have been obtained by Thierry Le Pollès in Laboratoire Ondes et Acoustique, ESPCI, Paris, and are the illustrations of his numerical results calculated in C++.

Remark 2 *We would like also illustrate the case of "quasi-shock" using [10, pp.78-81]. This phenomenon appears for the KZK equation with small viscosity coefficient. According to [10] the wave is named a quasi shock wave if the breadth of the wave front $\triangle\tau \leq \pi/10$. The figures 8 and 9 have been obtained in [10] for* *the following problem for the density function of a beam rotationally invariant around the x_1 axis (cf. (37))*

$$\frac{\partial^2\rho^2}{\partial\tau\partial z} - N\frac{\partial^2\rho^2}{\partial\tau^2} - \delta\frac{\partial^3\rho}{\partial\tau^3} - \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)\rho = 0, \tag{39}$$
$$\rho|_{z=0} = -e^{-r^2}\sin\tau.$$

Remark 3 *There are mathematical works [25], [26] for KZK type equation*

$$\alpha u_{z\tau} = (f(u_\tau))_\tau + \beta u_{\tau\tau\tau} + \gamma u_\tau + \triangle_x u,$$

where $u_\tau = u_\tau(z,x,\tau)$ is the acoustic pressure, $(z,x) \in \mathcal{R}^d \times \mathcal{R}$, $d = 1,2$ are space variables and τ is the retarded time. The equation is studied with the hypothesis that the nonlinearity f has bounded derivative which allows to proof the global existence for the case when the coefficients are rapidly oscillating functions of z. So this problem is not related with our "acoustical" problem for the KZK equation where as we will see later there is a blow-up result illustrating the existence of a shock wave.

2 Mathematical Studies of the Cauchy Problem for KZK Equation

2.1 Existence uniqueness and stability of solutions of the KZK equation

Following the mathematical tradition in this section and in the next one the unknown will be denoted by u, and the variables $(x,y) \in \mathcal{R}_x \times (\Omega \subseteq \mathcal{R}^{n-1})$. When $\Omega \neq \mathcal{R}^{n-1}$ it is assumed that the solution satisfies on its boundary the Neumann boundary condition. Multiplying u by a positive scalar one reduces the problem to an equation involving only two constants β and γ

$$(u_t - uu_x - \beta u_{xx})_x - \gamma\Delta_y u = 0 \quad \text{in } \mathcal{R}_x/(L\mathcal{Z}) \times \Omega. \tag{40}$$

For sake of simplicity and because this also corresponds to practical situations [10, 39] we consider solutions which are periodic with respect to the variable x and which are of mean value zero:

$$u(x+L,y,t) = u(x,y,t), \quad \int_0^L u(x,y,t)dx = 0. \tag{41}$$

Observe that the conditions (41) are compatible with the flow and that the second one is "natural" because we consider fluctuations.

For these functions the norm of the space H^s ($s \in R$, $s \geq 0$) is denoted by

$$\|u\|_{H^s} = \left(\int\limits_{R^{n-1}} \sum_{k=-\infty}^{+\infty} (1+k^2+\eta^2)^s |\hat{u}(k,\eta)|^2 d\eta\right)^{\frac{1}{2}}.$$

If we introduce the operator $\Lambda = (1-\Delta)^{\frac{1}{2}}$ as $\widehat{(\Lambda u)}(\zeta) = (1+|\zeta|^2)^{\frac{1}{2}}\hat{u}(\zeta)$, then

$$\Lambda^s = (1-\Delta)^{\frac{s}{2}}, \quad \|u\|_{H^s} = \|\Lambda^s u\|_{L_2}. \tag{42}$$

We define the inverse of the derivative ∂_x^{-1} as an operator acting in the space of periodic functions with mean value zero this gives the formula:

$$\partial_x^{-1} f = \int_0^x f(s)ds + \int_0^L \frac{s}{L} f(s)ds. \tag{43}$$

This form of the operator ∂_x^{-1} preserves the both qualities: the periodicity and having the mean value zero.

In this situation equation (40) is equivalent to the equation

$$u_t - uu_x - \beta u_{xx} - \gamma \partial_x^{-1} \Delta_y u = 0 \quad \text{ in } \mathcal{R}_x/(L\mathcal{Z}) \times \Omega. \tag{44}$$

Finally when $\gamma = 0$ equation (40) reduces to the Burgers-Hopf equation for which existence smoothness and uniqueness of solution are well known. For $\gamma = \beta = 0$ it reduces to the Burgers equation

$$\partial_t u - \partial_x \frac{u^2}{2} = 0,$$

which after a finite time exhibits singularities. After this "blow-up" time the solution can be uniquely continued into a weak solution satisfying an elementary entropy condition (in the present case with $\gamma \neq 0$ it seems that this construction cannot be adapted to equation (40) with $\beta = 0$ and $\gamma \neq 0$).

We would like also to notice that the J. Bourgain-type method and introduction the Bourgain spaces as in [35, 29, 30] and others are not useful for the KZK problem because of absence of the terms with an odd derivative as for example u_{xxx} in (44). The presence only of the second derivative make impossible the main estimations and equalities of this method.

2.1.1 A priori estimates for smooth solutions

According to the standard approach we first establish a priori estimates for smooth solutions which are in particular a consequence of the relation:

$$\begin{aligned} \int_0^L \int_{\mathcal{R}_y^{n-1}} \partial_x^{-1}(\Delta_y u) u dx dy &= -\int_0^L \int_{\mathcal{R}_y^{n-1}} \partial_x^{-1}(\nabla_y u) \nabla_y u dx dy \\ &= \int_0^L \int_{\mathcal{R}_y^{n-1}} \partial_x^{-1}(\nabla_y u) \partial_x (\partial_x^{-1}(\nabla_y u)) dx dy = 0. \end{aligned} \tag{45}$$

The L_2 norm and the H^s in $(\mathcal{R}_x^+/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$ are denoted by $|u|$ and by $\|u\|_s$.

Proposition 1 *The following estimates are valid for solutions of the integrated KZK equation (44):*

$$\frac{1}{2}\frac{d}{dt}|u(\cdot,\cdot,t)|^2 + \beta|\partial_x u(\cdot,\cdot,t)|^2 = 0\,, \tag{46}$$

$$\textit{For } s > [\frac{n}{2}]+1 \quad \frac{1}{2}\frac{d}{dt}\|u\|_s^2 + \beta\|\partial_x u\|_s^2 \le C(s)\|u\|_s^3 \tag{47}$$

$$\textit{and} \;\; \frac{1}{2}\frac{d}{dt}\|u\|_s^2 + \beta C(L)\|u\|_s^2 \le C(s)\|u\|_s^3. \tag{48}$$

The estimates (47), (48) are valid for $s > [\frac{n}{2}]+1$ which is the necessary condition because of application of the Sobolev theorem.

Proof. To obtain the relation (46) multiply (44) by u, and integrate by part. It shows that for $\beta = 0$ we have the conservation law for the norm of u in $L_2(\mathcal{R}_x^+/(L\mathcal{Z}))\times\mathcal{R}_y^{n-1})$. If $\beta > 0$ we also have according to the physical phenomena [10] the dissipation of energy.

For the clarity the proof of (47) is done firstly in 3 space variables, with $\Omega = \mathcal{R}^2$ and s an integer (i.e. in the present case $s = 3$) and after we give the proof in general case. In $2d$ in particular when $\Omega = S^1$ the proof is even simpler. The proof in the whole is similar except for the relation (48) which holds only in the periodic case and not on the whole line. (In this later case the H^s norm of $\partial_x u$ does not control the H^s norm of u).

For the proof of general case $s \in \mathcal{R}$ one has used the representation of the norm in H^s with the help of the operator Λ by (42) and the technique demonstrated in [23] and [34] for periodic and nonperiodic cases, which allows to deduce

$$\frac{1}{2}\frac{d}{dt}\|u\|_s^2 + \beta\|\partial_x u\|_s^2 \le C\|\nabla_{x,y}u\|_{L_\infty}\|u\|_s^2,$$

and this implies the necessity of our restriction for s:

$$\text{if } s > [\frac{n}{2}]+1 \text{ then } \; H^{s-1}\subset L_\infty.$$

The elementary proof

The introduction of the H^3 norm for $n = 3$ comes from the control of the nonlinearity with the Sobolev theorem. It starts with the estimating

$$\int_0^L\int_{\mathcal{R}_y^2}\partial_x^3(u\partial_x u)\partial_x^3 u dxdy. \tag{49}$$

We integrate it by parts, use the x periodicity, and finally one has:

$$|\int_0^L\int_{\mathcal{R}_y^2}\partial_x^3(u\partial_x u)\partial_x^3 u dxdy| \le C|\partial_x u|_{L^\infty(]0,L[,\times\mathcal{R}_y^2)}\|u\|^2_{H^3(\mathcal{R}_x\times\mathcal{R}_y^2)}.$$

In the same way with ∂_y denoting the derivative with respect to any orthogonal component one obtains:

$$\int_0^L\int_{\mathcal{R}_y^2}\partial_y^3(u\partial_x u)\partial_y^3 u dxdy=\int_0^L\int_{\mathcal{R}_y^2}u\partial_x(\partial_y^3 u)\partial_y^3 u dxdy+\int_0^L\int_{\mathcal{R}_y^2}(\partial_y^3 u)^2\partial_x u dxdy+$$
$$+3\int_0^L\int_{\mathcal{R}_y^2}(\partial_y u)(\partial_y^2\partial_x u)\partial_y^3 u dxdy+3\int_0^L\int_{\mathcal{R}_y^2}\partial_y^2 u(\partial_y\partial_x u)\partial_y^3 u dxdy \qquad (50)$$

and as above one has

$$\int_0^L\int_{\mathcal{R}_y^2}u\partial_x(\partial_y^3 u)\partial_y^3 u=-\frac{1}{2}\int_0^L\int_{\mathcal{R}_y^2}\partial_x u(\partial_y^3 u)^2 dxdy.$$

Therefore the sum of the first, second and last term of (50) are bounded by

$$C|\partial_y u|_{L^\infty(]0,L[\times\mathcal{R}_y^2)}\|u\|^2_{H^3(]0,L\times\mathcal{R}_y^2)}$$

and for the third term one can write:

$$\begin{aligned}\int_0^L\int_{\mathcal{R}_y^2}\partial_y^2 u(\partial_y\partial_x u)\partial_y^3 u dxdy &= \frac{1}{2}\int_0^L\int_{\mathcal{R}_y^2}\partial_y(\partial_y^2 u)^2(\partial_y\partial_x u)dxdy\\ &= -\frac{1}{2}\int_0^L\int_{\mathcal{R}_y^2}(\partial_y^2 u)^2\partial_x(\partial_y^2 u)dxdy=0.(51)\end{aligned}$$

Finally one has obtained the following estimate:

$$\begin{aligned}&\int_0^L\int_{\mathcal{R}_y^2}(\partial_x^3(u\partial_x u)(\partial_x^3 u)+\sum_{1\le i\le 2}\partial_{y_i}^3(u\partial_x u)(\partial_{y_i}^3 u))dxdy\le\\ &\le(\sup_{x,y}|\partial_x u(x,y,t)|+|\nabla_y u(x,y,t)|)\|u\|_3^2. \qquad (52)\end{aligned}$$

The choice of the index of derivation 3 comes from the Sobolev theorem which gives:

$$|\partial_x u|+|\partial_y u|\le C\|u\|_{H^3(]0,L[\times\mathcal{R}_y^2)}.$$

Eventually to obtain (47) write:

$$\begin{aligned}0 &= \int_0^L\int_{\mathcal{R}_y^2}[\partial_x^3(u_t-uu_x-\beta u_{xx}-\gamma\partial_x^{-1}\Delta_y u(s,y)ds).\partial_x^3 u\\ &+ \sum_{1\le i\le 2}\partial_{y_i}^3(u_t-uu_x-\beta u_{xx}-\gamma\partial_x^{-1}\Delta_y u(s,y)ds).\partial_{y_i}^3 u]dxdy\end{aligned}$$

and use the estimate (52). Finally to prove (48) one uses the fact that u is of x mean value 0 and therefore it is (cf: (43)) related to $\partial_x u$ by the formula

$$u=\partial_x^{-1}\partial_x u=\int_0^x\partial_x u(s,y)ds+\int_0^L\frac{s}{L}\partial_x u(s,y)ds, \qquad (53)$$

which implies the relation

$$\|u\|_{H^3(]0,L[\times\mathcal{R}_y^2)} \le C\|\partial_x u\|_{H^3(]0,L[\times\mathcal{R}_y^2)}\,.$$

The general proof

We apply the operator Λ^s to equation (44) and multiply by $\Lambda^s u$ in L_2

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^s}^2 - \beta(\Lambda^s u_{xx},\Lambda^s u) - (\Lambda^s(uu_x),\Lambda^s u) = 0,$$
$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^s}^2 + \beta\|u_x\|_{H^s}^2 - (\Lambda^s(uu_x),\Lambda^s u) = 0.$$

Suppose that $[\Lambda^s,u]v = \Lambda^s(uv) - u\Lambda^s v$. Then

$$(\Lambda^s(uu_x),\Lambda^s u) = ([\Lambda^s,u]u_x,\Lambda^s u) + (u\partial_x\Lambda^s u,\Lambda^s u) = ([\Lambda^s,u]u_x,\Lambda^s u) - \frac{1}{2}(u_x\Lambda^s u,\Lambda^s u).$$

As soon as $2(s-1) > n$, the last term is estimated by

$$|(u_x\Lambda^s u,\Lambda^s u)| \le \|u_x\|_{L_\infty}\|u\|_{H^s}^2 \le C\|u_x\|_{H^{s-1}}\|u\|_{H^s}^2 \le C\|u\|_{H^s}^3.$$

For the first term we have:

$$([\Lambda^s,u]u_x,\Lambda^s u) \le \|[\Lambda^s,u]u_x\|_{L_2}\|u\|_{H^s} \le C\|u\|_{H^s}\|u_x\|_{H^{s-1}}\|u\|_{H^s} \le C\|u\|_{H^s}^3.$$

We need now the following proposition.

Proposition 2 *With the above notations we have the estimate*

$$\|[\Lambda^s,u]u_x\|_{L_2} \le C\|u\|_{H^s}\|u_x\|_{H^{s-1}}.$$

Proof.

For the periodic case, using the result of J.C. Saut and R. Temam from [34] which consists in the following:

if u, v are in $H^s(\mathcal{R}^n)$ or in $H^s(\mathcal{R}^n/\mathcal{Z}^n)$ and $s\in\mathcal{R}$, $s>1$, $\gamma\in\mathcal{R}$, $\gamma>n/2$, then

$$\|D^s(uv) - uD^s v\|_{L_2} \le c(\gamma,s)\{\|u\|_s\|v\|_\gamma + |u\|_{\gamma+1}\|v\|_{s-1}\},$$

what is easy to generalize for $H^s(\mathcal{R}_x/(L\mathcal{Z}))\times\mathcal{R}_y^{n-1}$. The estimate remains true if we change D^s on Λ^s.

In our case $\gamma = s-1$, from where the result follows. $\square$

To finish the proof for (48) we notice that $\|u\|_{H^s} \le C\|\frac{\partial u}{\partial x}\|_{H^s}$, because of (53). $\square$

2.1.2 Existence and uniqueness for smooth solutions

The following theorem is an easy consequence of the a priori estimates.

Theorem 2 *For the following Cauchy problem*

$$u_t - uu_x - \beta u_{xx} - \gamma \partial_x^{-1}(\Delta_y u) = 0\,, u(x,y,0) = u_0 \tag{54}$$

considered in $(\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1}$, *i.e. in the class of* x *periodic functions with mean value* 0 *with the operator* ∂_x^{-1} *defined by the formula (43) and finally with* $\beta \geq 0$ *one has the following results.*

1 For $s > [\frac{n}{2}] + 1$ *(*$s = 3$ *for instance in dimension* 3*) there exists a constant* $C(s,L)$ *such that for any initial data* $u_0 \in H^s$ *the problem (54) has on an interval* $[0, T[$ *with*

$$T \geq \frac{1}{C(s,L)\|u_0\|_{H^s}} \tag{55}$$

a solution in $C([0,T[, H^s) \cap C^1([0,T[, H^{s-2})$.

2 Let T^* *be the biggest time on which such solution is defined then one has*

$$\int_0^{T^*} \sup_{x,y}(|\partial_x u(x,y,t)| + |\nabla_y u(x,y,t)|)dt = \infty. \tag{56}$$

3 If $\beta > 0$ *there exists a constant* C_1 *such that*

$$\|u_0\|_s \leq C_1 \Rightarrow T^* = \infty. \tag{57}$$

4 For two solutions u *and* v *of KZK equation, assume that* $u \in L_\infty([0,T[; H^s)$, $v \in L^2([0,T[; L_2)$. *Then one has the following stability uniqueness result:*

$$|u(,t) - v(.,t)|_{L^2} \leq e^{\int_0^t sup_{x,y}|\partial_x u(x,y,s)|ds}|u(.,0) - v(.,0)|_{L^2}. \tag{58}$$

Remark 4 *The estimate (58) is of strong-weak form, as in [14] only the* L_∞ *norm of* u_x *is needed.*

Remark 5 *When there is no viscosity all the corresponding statements of the theorem 2 remain valid for* $0 > t > -C$ *with a convenient* C*.*

Remark 6 *As (40) is envisaged for* $u(t,x,y)$ *with* $x \in \mathcal{R}/(L\mathcal{Z})$*, the KZK equation can be also written for* $u(t,x,y) = v(t,-x,y)$ *in the equivalent form*

$$(v_t + vv_x - \beta v_{xx})_x + \gamma \triangle_y v = 0.$$

So it is important to keep invariant the sign $-\beta v_{xxx}$, $\beta \geq 0$*, but all other signs can be changed.*

Proof. To construct a solution one can proceed by regularization, by a fractional step method, or by any other type of approximation. In particular it was done for the general case with the help of Kato theory from [19, 20, 21, 22]. Since we intend to analyze the numerical methods, the fractional step is favored and once again the only case $n = 3$ and $s = 3$ with periodic solutions is analyzed. The idea of this kind of proof can be found in [37] and firstly have been introduced by Marchuk and Yanenko. Furthermore as for a priori estimates result we cite two proofs: one with the analysis of the fractional step method for the case $n = 3$ and $s = 3$ and an other proof for general case.

The application of the fractional step method

To control the stability of the fractional step method one uses the following

Lemma 1 *Let $X_0\,, C\,, T$ be three positive numbers with*

$$T < \frac{2}{C\sqrt{X_0}}.$$

Let N be a positive integer, $\Delta T = \frac{T}{N}$ and for $0 \le k \le N$ let X_k be a sequence of positive numbers which satisfy the estimate:

$$\text{for } 0 \le k \le N-1\,, \quad X_{k+1} \le \frac{X_k}{(1-\frac{1}{2}C\Delta T\sqrt{X_k})^2},$$

then for any $0 \le k \le N$ one has

$$X_k \le \frac{X_0}{(1-\frac{1}{2}CT\sqrt{X_0})^2}.$$

Proof. The solution of the equation:

$$y' = Cy^{\frac{3}{2}}\,, y(0) = X_0$$

is given by the formula:

$$y(t) = \frac{X_0}{(1-\frac{1}{2}Ct\sqrt{X_0})^2}$$

and is therefore positive and bounded on the interval $[0, T]$. Denote by y_k the value of this solution at the points $k\Delta T$ they satisfy the relation

$$y_{k+1} = \frac{y_k}{(1-\frac{1}{2}C\Delta T\sqrt{y_k})^2}$$

and therefore for any $k \in [0, N-1]$ one has

$$0 \le X_k \le y_k$$

and the conclusion follows.

The operator $\partial_x^{-1}\Delta_y$ is the generator of a unitary group in the space of $L^2(\frac{\mathcal{R}}{\mathcal{Z}L}\times\Omega)$ with mean value zero and this unitary group

$$e^{-t\partial_x^{-1}\Delta_y}$$

preserves the H^s norm. In the mean time the solution of the Burgers equation:

$$\partial_t u - uu_x - \beta u_{xx} = 0$$

on the time interval $]k\Delta T,(k+1)\Delta T[$ with u given at the time $k\Delta T$ may increase (as in the proof of the a priori estimates) the H^3 according to the formula

$$\|u((k+1)\Delta T)\|_3 \le \frac{\|u(k\Delta T)\|_3}{(1-\frac{1}{2}CT\sqrt{\|u(k\Delta T)\|_3})^2}.$$

According to the tradition one defines on the interval $[0,T]$ the functions u_N and $u_{N+\frac{1}{2}}$ by the following formula:

$$\begin{aligned}
&\text{for } t\in]k\Delta T,(k+1)\Delta T[\,, \quad u_{N+\frac{1}{2}}(0_-)=u_0,\\
&\partial_t u_N - u_N(u_N)_x - \beta(u_N)_{xx}=0,\ u_N(k\Delta T)=u_{N+\frac{1}{2}}(k\Delta T_-),\\
&\partial_t u_{N+\frac{1}{2}} - \gamma\partial_x^{-1}\Delta_y u_{N+\frac{1}{2}}=0,\ u_{N+\frac{1}{2}}(k\Delta T)=u_N((k+1)\Delta T_-).
\end{aligned}$$

The lemma 1 implies that the functions u_N and $u_{N+\frac{1}{2}}$ are uniformly bounded in

$$L^\infty(]0,T[,H^3)$$

and by a standard argument as it is done for instance in [37] they converge in

$$C(]0,T[,H^2)$$

to a function u which is solution of the KZK equation:

$$\partial_t u - uu_x - \beta u_{xx} - \gamma\partial_x^{-1}\Delta_y u = 0.$$

The fact that the solution $u\in C([0,T[,H^s)\cap C^1([0,T[,H^{s-2})$ can be easily shown as in [19].

This proof being invariant with respect to time translation shows also that whenever $u(t)\in H^s$ is finite the solution can be extended on a non zero time interval which is bounded below in term of $\|u(t)\|_s$. Now from the estimate (52) one deduces the relation:

$$\|u(t_2)\|_s^2 \le 2\|u(t_1)\|_s^2 e^{\int_{t_1}^{t_2}\sup_{x,y}(|\partial_x u(x,y,s)|+|\nabla_y u(x,y,s)|)ds} \tag{59}$$

and this proves point 2.

To prove the next point one observes that periodic solutions with mean value 0 satisfies for t small enough the estimate:

$$\frac{1}{2}\frac{d}{dt}\|u\|_s^2 + \|u\|_s^2(\beta C(L) - C(s)\|u\|_s) \le 0. \tag{60}$$

Therefore if for $t = 0$ one has

$$\beta C(L) - C(s)\|u(0,\cdot)\|_s \geq 0 \quad \text{i.e.} \quad \|u(0,\cdot)\|_s \leq \frac{\beta C(L)}{C(s)}$$

the quantity $\|u(t,\cdot)\|_s^2$ will decay for $t > 0$ and therefore satisfies the same estimate on all the interval $[0, T^*[$, which and therefore can be extended after any finite value T^* and this proves point 3. Finally let u and v be two solutions. For the difference one has the relation:

$$\partial_t(u-v) - (u-v)\partial_x u + v\partial_x(v-u) - \beta\partial_x^2(u-v) - \gamma\partial_x^{-1}\Delta_y(u-v) = 0. \quad (61)$$

Multiplying this equation by $(u-v)$ integrating in x and y and performing standard integration by parts gives:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|u-v|^2 - \int_0^L \int_{\mathcal{R}_y^{n-1}} \partial_x u(u-v)^2 dxdy + \\ &+ \int_0^L \int_{\mathcal{R}_y^{n-1}} v(u-v)\partial_x(u-v) dxdy + \\ &+\beta \int_0^L \int_{\mathcal{R}_y^{n-1}} (\partial_x(u-v))^2 dxdy = 0, \end{aligned} \quad (62)$$

which, with the relation:

$$\begin{aligned} &\int_0^L \int_{\mathcal{R}_y^{n-1}} v(u-v)\partial_x(u-v) dxdy = \int_0^L \int_{\mathcal{R}_y^{n-1}} [(v-u)+u]\frac{\partial_x(u-v)^2}{2} dxdy = \\ &= -\frac{1}{2}\int_0^L \int_{\mathcal{R}_y^{n-1}} \partial_x u\,(u-v)^2 dxdy \end{aligned} \quad (63)$$

leads to the estimate

$$\frac{1}{2}\frac{d}{dt}|u-v|_{L_2}^2 \leq \sup_{x,y} |\partial_x u(x,y,t)|\, |u-v|_{L_2}^2, \quad (64)$$

and the Gronwall lemma gives (58).

2.1.3 General proof of the existence theorem

All functions in this part are supposed to have mean value zero:

$$\int_0^L u dx = 0.$$

The theory of quasilinear evolution equations necessary to our proof can be found in [19, 20, 21, 22].

An abstract problem associated to the KZK equation

For $s > [\frac{n}{2}] + 1$ we study the following problem in the Sobolev space $H^{s-2}((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$ of zero mean valued functions

$$\frac{du}{dt} + A(u)u = 0, \tag{65}$$

$$u(0) = u_0,$$

where $A(u)v = -\beta D_x^2 v - \gamma K v - u D_x v$ and K is defined by

$$\mathcal{F}(Ku)(m,\eta) = \begin{cases} \frac{-L\eta^2}{i2\pi m}\hat{u}(m,\eta), & \text{if } m \neq 0 \\ 0, & \text{if } m = 0. \end{cases} \tag{66}$$

The definition (43) of the operator ∂_x^{-1} which preserves the periodicity and having the mean value zero of considered functions will be reformulated now in terms of Fourier transform:

$$\partial_x^{-1} f = \sum_{k \neq 0} \frac{\widehat{f(k)}}{2\pi i \frac{k}{L}} e^{2i\pi\frac{kx}{L}}.$$

Besides, the function $f(x,y)$ is periodic on x and of mean value zero if and only if $\widehat{f}(0,\xi) = 0$ for all $\xi \in \mathcal{R}$.

The local existence

By using the method of proof from [20] (see also [19]), we obtain the existence of the KZK solution.

More precisely we obtain

Theorem 3 *Let $s > [\frac{n}{2}] + 1$ and let $u_0 \in H^s((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$ (periodic in x with mean value zero). Then there exists $T > 0$, which depends only on $\|u_0\|_{H^s}$, such that the problem*

$$u_t - uD_x u - \beta D_x^2 u - \gamma \int_0^x \triangle_y u ds + \int_0^L \frac{s}{L} \triangle_y u ds = 0, \tag{67}$$
$$u|_{t=0} = u_0$$

has a unique non-continuable solution

$$u \in C([0,T), H^s((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})) \cap C^1([0,T), L_2((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1}))$$

(periodic in x with mean value zero). Besides, if $0 < \bar{T} < T$, then the solution $u \in C([0,\bar{T}], H^s((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})) \cap C^1([0,\bar{T}], L_2((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1}))$ depends continuously on the initial value u_0, i.e., the mapping $\tilde{u}_0 \mapsto \tilde{u}$ is continuous from $H^s((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$ to $C([0,\bar{T}]; H^s((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1}))$.

Corollary 1 *Giving* $u_0 \in H^s((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$, $s > [\frac{n}{2}] + 1$, *there exists* $T > 0$ *and a unique function* $u \in C([0,T), H^s((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})) \cap C^1([0,T), L_2((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1}))$ *such that*

$$(u_t - uD_xu - \beta D_x^2 u)_x - \gamma \Delta_y u = 0, \tag{68}$$
$$u|_{t=0} = u_0.$$

The solution depends continuously on the initial value u_0.

Proof. It is easy to verify that the solution of (67) is a solution of (68).

On the other hand, if $u \in C([0,T), H^s) \cap C^1([0,T), L_2)$ is a solution of (68), then

$$(u_t - uD_xu - \beta D_x^2 u)_x = \gamma \Delta_y u \in C([0,T); H^{s-2}) \hookrightarrow C([0,T); L_2).$$

Hence $u_t - uD_xu - \beta D_x^2 u \in H_x^1$, and

$$i\frac{2\pi}{L} m\mathcal{F}(u_t - \beta D_x^2 u - uD_xu)(m,\xi) = \mathcal{F}[(u_t - \beta D_x^2 u - uD_xu)](m,\xi) =$$
$$= \gamma \mathcal{F}(\triangle_y u)(m,\xi) = -\gamma \xi^2 \widehat{u}(m,\xi).$$

Thus, for $m \neq 0$,

$$\widehat{(u_t)}(m,\xi) + i\frac{2\pi}{L} m \left(\beta \frac{4\pi^2}{L^2} m^2 + \gamma \frac{L\xi^2}{i2\pi m}\right) \widehat{u}(m,\xi) - (\widehat{uD_xu})(m,\xi) = 0,$$

which means that, as from definition of the operator $\check{A}$ for $\check{A}(u)u \int_0^L uD_xu dx = 0$,

$$(\widehat{u_t})(m,\xi) + (\widehat{\tilde{A}(u)})(m,\xi) + (\widehat{\check{A}(u),u})(m,\xi) = 0,$$

equality which is also valid, in a trivial way, for $m = 0$. Therefore $u_t + A(u)u = 0$, and thus u is the solution of (67), which implies that the solution of (68) is unique.

Remark 7 *It may be seen from the equation that the solution* u *(periodic in* x *with mean value zero) belongs to* $C^1([0,T), H^{s-2}((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1}))$.

The following theorem which can be proved in exactly the same way as theorem 2.3 from [19, p. 573], shows that $\bar{T}$ does not depend on s.

Theorem 4 (Regularity) *If* $u \in C([0,\bar{T}], H^s(\Omega)) \cap C^1([0,\bar{T}], L_2(\Omega))$ *is a solution of problem (68) and* $u_0 \in H^{s'}$ *with* $s' > s > [\frac{n}{2}] + 1$ *(for periodic on* x *mean value functions), then* $u \in C([0,\bar{T}], H^{s'}) \cap C^1([0,\bar{T}], H^{s'-2})$ *with the same* $\bar{T}$.

Remark 8 *For nonperiodic case the local existence can be easily proved using the estimate (47) in the form*

$$\|u\|_s^2 \leq C(s) \|u\|_s^3$$

(to prove it we can use the estimation from [23]) and using the technique of [19] with regularization of system (65):

$$\frac{du}{dt} + A_\varepsilon(u)u = 0,$$

$$u(0) = u_0.$$

Here $A_\varepsilon(u)v = -D_x^2 v - K_\varepsilon v - uD_x v$ and K_ε is defined by

$$\mathcal{F}(K_\varepsilon u)(m,\xi) = \frac{-2\pi m \xi^2}{iL(\varepsilon + \frac{4\pi^2}{L^2}m^2)}\hat{u}(m,\xi).$$

By using the method of the proof from [19], we obtain the solution of KZK equation passing to the limit $\varepsilon \to 0$.

This regularization have been also done in [38].

Global existence in time of the solution for rather small initial data

Now let us prove that the maximal time of existence $T = \infty$ for rather small initial data.

Lemma 2 *For all $t \in [0,T)$,*

$$\|u\|_{H^s}^2 + \beta C_1(L)\int_0^t \|u\|_{H^s}^2 d\tau \le C_2(s)\int_0^t \|u\|_{H^s}^3 d\tau + \|u_0\|_{H^s}^2, \tag{69}$$

where $\beta C_1(L)$ and $C_2(s)$ are positive constants. In particular, for all initial data u_0 satisfying

$$\|u_0\|_{H^s} \le \frac{\beta C_1(L)}{C_2(s)},$$

the time of existence of the solution is $T = +\infty$ and

$$\|u\|_{C([0,+\infty),H^s)} \le \frac{\beta C_1(L)}{C_2(s)}. \tag{70}$$

Proof. By using the regularity theorem for $u_0 \in H^{s+2}$, we have $u \in C([0,T), H^{s+2})$, and, thus, we apply the operator Λ^s to the equation

$$u'(t) + A(u(t))u(t) = 0, \quad t \in [0,T),$$

and take the inner product in L_2 with $\Lambda^s u$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^s}^2 - (\Lambda^s(u_{xx}),\Lambda^s u) - (\Lambda^s(uu_x),\Lambda^s u) + \int \sum_m (1+\frac{4\pi^2}{L^2}m^2+\xi^2)^s \frac{L\xi^2}{i2\pi m}|\hat{u}(m,\xi)|^2 d\xi = 0.$$

Taking the real part of the former expression, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_{H^s}^2 + \|u_x\|_{H^s}^2 - (\Lambda^s(uu_x),\Lambda^s u) = 0.$$

Using now the proof of proposition 1 we obtain (69).

We define now $y(t)=\|u\|_{H^s}$, such that $y(0)=\|u_0\|_{H^s}$, thus we obtain the equation

$$\frac{d}{dt}(y^2)=C_2(s)y^3-\beta C_1(L)y^2.$$

Solving it we find that

$$y(t)=\left(\frac{C_2(s)}{\beta C_1(L)}-\left(\frac{C_2(s)}{\beta C_1(L)}-\frac{1}{\|u_0\|}\right)e^{\frac{\beta C_1(L)}{2}t}\right)^{-1},$$

from where, imposing $\|u_0\|_{H^s}\leq\frac{\beta C_1(L)}{C_2(s)}$, we obtain that $T=+\infty$ and it follows that

$$\|u(t)\|_{H^s}\leq y(t)\leq\frac{\beta C_1(L)}{C_2(s)}\quad\forall t\in[0,+\infty).$$

Lemma 3 *Let $s>[\frac{n}{2}]+1$ and suppose $u_0\in H^s((\mathcal{R}/(L\mathcal{Z}))\times\mathcal{R}_y^{n-1})$ is such that $\partial_x^{-1}\triangle_y u_0=\phi_0\in H^{s-2}$ and $\|u_0\|_{H^s}\leq\frac{\beta C_1(L)}{C_2(s)}$. Then there exists a constant C such that*

$$\|u'(t)\|_{C([0,+\infty),H^{s-2})}\leq C. \tag{71}$$

Proof. For $t\in[0,+\infty)$ and $h>0$ sufficiently small, let $z(t)=h^{-1}[u(t+h)-u(t)]$. Then, having subtracted the KZK equation for $u(t)$ from the KZK equation for $u(t+h)$ and having divided by h, we obtain

$$z'(t)-D_x^2z(t)-Kz(t)-u(t)D_xz(t)=D_xu(t+h)z(t).$$

Let $l=s-2\geq 0$. Applying the operator Λ^l to the above equation, taking the inner product in L_2 with $\Lambda^l z(t)$, integrating by parts, and considering only real parts we obtain

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|^2_{H^l}+\|D_xz(t)\|^2_{H^l}-(\Lambda^l(u(t)D_xz(t)),\Lambda^lz(t))=(\Lambda^lf(t),\Lambda^lz(t)),$$

where $f(t)=D_xu(t+h)z(t)$. Thanks to [19, p.576, 577] one has the estimates

$$|(\Lambda^l(uD_xz),\Lambda^lz)|\leq C\|u\|_{H^s}\|z\|^2_{H^l}\quad\forall l>0,$$

$$|(\Lambda^lf,\Lambda^lz)|\leq C\|u(t+h)\|_{H^s}\|z\|^2_{H^l}\quad\forall l>0,$$

which with $-\|D_xz(t)\|_{H^l}\leq-\beta C_1(L)\|z(t)\|_{H^l}$ give

$$\frac{1}{2}\frac{d}{dt}\|z(t)\|^2_{H^l}\leq C(\|u(t+h)\|_{H^s}+\|u(t)\|_{H^s})\|z(t)\|^2_{H^l}-\beta C_1(L)\|z(t)\|^2_{H^l}\leq$$

$$\leq(2C\frac{\beta C_1(L)}{C_2(s)}-\beta C_1(L))\|z(t)\|^2_{H^l}=\beta C_1(L)\left(\frac{2C}{C_2(s)}-1\right)\|z(t)\|^2_{H^l},$$

from where, as the coefficient $\frac{2C}{C_2(s)}-1$ can be, thanks to the choice of constants, negative, as well as positive or zero, we obtain that

$$\frac{d}{dt}\|z(t)\|^2_{H^l}\leq 0,$$

which implies using the Gronwall lemma that

$$\|z(t)\|_{H^l} \leq \|z(0)\|_{H^l}.$$

Passing to the limit for $h \to 0^+$ we find that

$$\|u'(t)\|_{H^l} \leq \|u'(0)\|_{H^l}.$$

But

$$\|u'(0)\|_{H^l} \leq \|D_x^2 u_0 + u_0 D_x u_0\|_{H^l} + \|Ku_0\|_{H^l},$$

and

$$\|Ku_0\|^2_{H^l} = \int\limits_{\mathbb{R}^{n-1}} \sum_m (1 + \frac{4\pi^2}{L^2} m^2 + \xi^2)^l \left| \frac{L\xi^2}{2\pi m} \hat{u}_0 \right|^2 d\xi =$$

$$= \int\limits_{\mathbb{R}^{n-1}} \sum_m (1 + \frac{4\pi^2}{L^2} m^2 + \xi^2)^l \left| \frac{L}{2\pi m} \widehat{D_x \phi_0} \right|^2 d\xi =$$

$$= \int\limits_{\mathbb{R}^{n-1}} \sum_m (1 + \frac{4\pi^2}{L^2} m^2 + \xi)^l \left| \frac{i2\pi Lm}{2\pi Lm} \right|^2 |\hat{\phi}_0|^2 d\xi \leq \|\phi_0\|^2_{H^l}.$$

From where (71) follows. $\square$

This concludes the proof in general case of theorem 2 and we can reformulate our result by the following theorem.

Theorem 5 *Let $u_0 \in H^s((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$, $s > [\frac{n}{2}] + 1$, periodic in x with mean value zero, such that $D_x^{-1} \triangle_y u_0 \in H^{s-2}((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$, i.e., there exists $\varphi_0 \in H^{s-2}((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1})$ with $D_x \varphi_0 = \triangle_y u_0$, and the norm $\|u_0\|_{H^s} \leq \frac{\beta C_1(L)}{C_2(s)}$ is rather small. Then there exists a unique global solution of the problem*

$$(u_t - u_{xx} - uu_x)_x - \triangle_y u = 0,$$
$$u(0) = u_0$$

$u \in C([0, +\infty), H^s((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1}))$ with $u' \overset{note}{=} du/dt \in L_\infty([0, +\infty), H^{s-2}((\mathcal{R}_x/(L\mathcal{Z})) \times \mathcal{R}_y^{n-1}))$.

2.2 Blow-up and singularities

The first remark is that for $\nu = 0$ (or $\beta = 0$) and for function independent of y the KZK equation

$$(u_t - uu_x - \beta u_{xx})_x - \gamma \Delta_y u = 0 \quad \text{ in } \mathcal{R}_{t_+} \times \mathcal{R}_x \times \Omega \tag{72}$$

becomes Burgers equation which is known to exhibit singularities. On the other hand the derivation and the approximation results of the following section show that any solution of the KZK equation has in its neighborhood a solution of the isentropic Euler equation. Once again it is known that such solution even with smooth initial data may exhibit singularities (cf. [14] or [36]). These observations are reflected by the fact that for $\beta = 0$ and $\gamma > 0$ the equation (72) may generate singularities.

We prove the geometric blow-up result using the method of S. Alinhac, which is based on the fact that the studied equation degenerates to the Burgers equation. In fact Alinhac's method is the generalized method of characteristics for the Burgers equation adapted to the multidimensional case. As we can see the equation (72) possess all this main properties, and gives us the reason to apply it.

For instance one has the theorem:

Theorem 6 *The equation*

$$(u_t - uu_x)_x - \gamma \Delta_y u = 0 \qquad \text{in } \mathcal{R}_{t_+} \times \mathcal{R}_x \times \Omega \tag{73}$$

with Neumann boundary condition on $\partial\Omega$ has no global in time smooth solution if

$$\sup_{x,y} \partial_x u(x, y, 0)$$

is large enough with respect to γ.

Remark 9 *As we can see from [10] the result of the theorem perfectly confirms the numerical results. In practically from figures 8 and 9 one observes that for $\beta \to 0$ the KZK equation has a quasi shock approaching to the shock wave, into which it degenerates for $\beta = 0$.*

Proof. The proof follows the ideas of S. Alinhac ([2], [3] and [4]). First the blow-up is observed for $\gamma = 0$ and related to a singularity in the projection of an unfolded "blow-up system". Second the properties of this unfolded blow-up system are shown to be stable under small perturbations. One uses a Nash-Moser theorem with tamed estimates and this is the reason why will exists a T^* such that:

$$\lim_{t \to T^*} (T^* - t) \sup_{x,y} \partial_x u(x, y, t) > 0.$$

Remark 10 *The Nash-Moser theory and the definition of the tamed estimates can be found in [7].*

Remark 11 *An equation of the type (73) is introduced by Alinhac to analyze the blow-up of multidimensional (in $\mathcal{R}^{2+1}$) nonlinear wave equation by following the wave cone*

$$\partial_t^2 u - \triangle_x u + \sum_{0 \le i,j,k \le 2} g_{ij}^k \partial_k u \partial_{ij}^2 u = 0,$$

where

$$x_0 = t, \quad x = (x_1, x_2), \quad g_{ij}^k = g_{ji}^k,$$

with small smooth initial data (see [5]). In fact this corresponds to the same scaling as the KZK equation because from this wave equation with some changes of variable and approximate manipulations Alinhac obtains (see [3, 5, 6])

$$\partial_{xt}^2 u + (\partial_x u)(\partial_x^2 u) + \epsilon \partial_y^2 u = 0.$$

Moreover, the Euler system, with $\rho = \rho_0 + \tilde{\rho}$ *and* $\nabla q(\rho) = \frac{1}{\rho} \nabla p(\rho)$*, can be written as*

$$\begin{aligned} &\partial_t \tilde{\rho} + \nabla.u + u \nabla \tilde{\rho} + \tilde{\rho} \nabla.u = 0, \\ &\partial_t u + q'(\rho_0) \nabla \tilde{\rho} + (u, \nabla) u + q'(\tilde{\rho}) \tilde{\rho} \nabla \tilde{\rho} = 0, \end{aligned}$$

or

$$\begin{aligned} &\partial_t \tilde{\rho} + \nabla.u = F(u, \nabla.u, \tilde{\rho}, \nabla \tilde{\rho}), \\ &\partial_t u + q'(\rho_0) \nabla \tilde{\rho} = G(u, \nabla.u, \tilde{\rho}, \nabla \tilde{\rho}). \end{aligned}$$

If we now derive the first equation on t *and take* ∇ *of the last, then we take their difference, we obtain the wave equation of Alinhac's form :*

$$\partial_t^2 \tilde{\rho} - q'(\rho_0) \triangle \tilde{\rho} + (\nabla G - \partial_t F)(u, \nabla.u, \tilde{\rho}, \nabla \tilde{\rho}) = 0.$$

The similar wave equation can be also obtained for u*.*

This is the reason for the analogy.

More precisely for a "beam" (rotationally invariant around the x axis) in 3 space variables the KZK equation has the form

$$\tilde{u}_{xt} - (\tilde{u}\tilde{u}_x)_x - \gamma \frac{1}{y} \tilde{u}_y - \gamma \tilde{u}_{yy} = 0, \tag{74}$$

$$\tilde{u}|_{t=0} = u_0(x, y), \quad \tilde{u}|_{x=K} = 0.$$

If we consider the KZK equation in $\mathcal{R}^3$ with $y = (y_1, y_2)$ for general case, the term $\frac{1}{y} \tilde{u}_y$ must be omitted and ∂_y be replaced by ∇_y.

Let $A_0 > 0$ be a fixed constant and $u_0 \in C^\infty$ be a function of variables x, y defined in the domain

$$\{(x, y) | x \in [-A_0, K], \quad y \in [r_0, r_1], \quad r_0 > 0\}.$$

For the reason of technical simplification, we assume

$$u_0(K, y) = \partial_x u_0(K, y) = 0.$$

Let the function $\partial_x u_0$ have on $[-A_0, K] \times [r_0, r_1]$ $(r_0 > 0)$ a unique positive maximum in the point

$$\begin{gathered} m_0 = (x_0, y_0), \quad -A_0 < x_0 < K, \quad \textit{such that} \quad \partial_x u_0(m_0) > 0, \\ \nabla_{x,y}(\partial_x u_0)(m_0) = 0, \quad \nabla_{x,y}^2 (\partial_x u_0)(m_0) \ll 0. \end{gathered} \tag{75}$$

The condition (75) is the necessary condition for geometric blow-up.

For a $\bar{T} > 0$, which is the blow-up time and is unknown, and a function $A_0(y,t) > 0$ to be specified (with $A_0(y,0) = A_0$), we have in the domain

$$D = \{(x,y,t)|x \in [-A_0(y,t), K], \quad y \in [r_0, r_1], \quad r_0 > 0, \quad t \in [0, \bar{T}[\} \qquad (76)$$

a free boundary problem.

Let in (74) $\tilde{u} = u_x$, then

$$u_{xxt} - (u_x u_{xx})_x - \gamma \frac{1}{y} u_{yx} - \gamma u_{yyx} = 0,$$

and so we have

$$L(u) \equiv u_{xt} - u_x u_{xx} - \gamma \frac{1}{y} u_y - \gamma u_{yy} = 0, \qquad (77)$$

$$u|_{t=0} = \partial_x^{-1} u_0, \quad u|_{x=K} = 0, \quad u_x|_{x=K} = 0.$$

The change of variables $\Phi : (s, Y, T) \to (x = \varphi(s, Y, T), y = Y, t = T)$, where $\varphi(s, y, t)$ is some unknown function such that $\partial_s \varphi > 0$, $\varphi|_{t=0} = s$, $\varphi|_{s=K} = 0$, allows to construct a blow-up system if we set

$$w(s,y,t) = u(\varphi(s,y,t),y,t), \quad v(s,y,t) = u_x(\varphi(s,y,t),y,t). \qquad (78)$$

The method of introducing $x = \varphi(s, y, t)$ is based on the method of characteristics which naturally appears for Burgers' equation

$$(\partial_t - u\partial_x)u = 0, \quad u|_{t=0} = u_0(x). \qquad (79)$$

Indeed, $x = \varphi(s,t) = s - tu_0(s)$, and the Cauchy problem

$$x_{tt} = 0, \quad x_t|_{t=0} = -u_0(s), \quad x|_{t=0} = s$$

with notation $x_t = \varphi_t = -u(\varphi(s,t),t) = -v$ becomes the blow-up system

$$v_t = 0, \quad v = -\varphi_t, \quad \varphi|_{t=0} = s, \quad v|_{t=0} = u_0. \qquad (80)$$

Since $v = u(\varphi(s,t),t)$ and $v_s = u_x \varphi_s$, in the case where the solutions of (80) satisfy the conditions $v_s \neq 0$, $\varphi_s = 0$ in some point, $u_x = v_s/\varphi_s$ becomes infinite. According to the terminology of Alinhac, the solution u displays geometric blow-up, because the blow-up of u does not come from the blow-up of v, but from the singularity of the change of variables Φ.

From (78) we have

$$w_s = u_\varphi \varphi_s|_{\varphi=x} = u_x \varphi_s = v\varphi_s,$$

$$A \equiv w_s - v\varphi_s = 0. \qquad (81)$$

Lets compute the derivatives in new variables which are present in (77)

$$u_{xx} = v_s(\varphi_s)^{-1}, \quad u_{xt} = v_t - v_s(\varphi_s)^{-1}\varphi_t,$$

$$u_y = w_y - v\varphi_y, \quad u_{yy} = w_{yy} - 2v_y\varphi_y + v_s(\varphi_s)^{-1}\varphi_y^2 - v\varphi_{yy}.$$

So we obtain

$$v_t + \frac{v_s}{\varphi_s}\left(-\varphi_t - v - \gamma\varphi_y^2\right) - \gamma(w_{yy} - 2v_y\varphi_y - v\varphi_{yy}) - \frac{\gamma}{y}(w_y - v\varphi_y) = 0,$$

$$L(u)(\varphi(s,y,t),y,t) = \mathcal{E}\frac{v_s}{\varphi_s} + \mathcal{R},$$

where

$$\mathcal{E} = -\varphi_t - v - \gamma\varphi_y^2, \tag{82}$$

$$\mathcal{R} = v_t - \gamma(w_{yy} - 2v_y\varphi_y - v\varphi_{yy}) - \frac{\gamma}{y}(w_y - v\varphi_y). \tag{83}$$

In this case the blow-up system is

$$\mathcal{A} = 0, \quad \mathcal{E} = 0, \quad \mathcal{R} = 0, \tag{84}$$

$$v|_{t=0} = u_0, \quad w|_{t=0} = \partial_s^{-1}u_0, \quad \varphi|_{t=0} = s, \quad w|_{s=K} = v|_{s=K} = \varphi|_{s=K} = 0.$$

From (78) it is easy to see that if we find the smooth solution of the blow-up system (84) such that in some point $\varphi_s = 0$, then the function u_{xx} has blow-up in this point which corresponds to the blow-up of $\tilde{u}_x$ of KZK equation's solution.

According to the change of variables we obtain that if the function $A_0(y,t)$ in definition of the domain D (76) is

$$A_0(y,t) = -\varphi(-A_0, y, t),$$

then

$$D_b = \{(s,y,t) | s \in [-A_0, K], \quad y \in [r_0, r_1], \quad r_0 > 0, \quad t \in [0, \bar{T}[\}.$$

In the domain D_b the blow-up system (84) has the form

$$\begin{cases} w_s - v\varphi_s = 0, \\ -\varphi_\tau - (v + \gamma\varphi_y^2) = 0, \\ v_\tau - \gamma(w_{yy} - 2v_y\varphi_y - v\varphi_{yy}) - \frac{\gamma}{y}(w_y - v\varphi_y) = 0, \\ v|_{\tau=0} = u_0, \quad w|_{\tau=0} = \partial_s^{-1}u_0, \quad \varphi|_{\tau=0} = s, \\ w|_{s=K} = v|_{s=K} = \varphi|_{s=K} = 0. \end{cases} \tag{85}$$

Note that for $\gamma = 0$ the problem (77) becomes the Burgers equation for $u_1 = u_x$ with the initial condition $u_1|_{t=0} = \partial_x\partial_x^{-1}u_0 = u_0$. This problem has a unique solution of C^∞ in D with the blow-up time

$$\bar{T} = \bar{T}_0 = \left(\sup_{x,y} \partial_x u_0\right)^{-1}. \tag{86}$$

In this point we have

$$\partial_x u = \frac{\partial_x u_0(x_0,y_0)}{1 - \bar{T}_0\partial_x u_0(x_0,y_0)} = \infty.$$

The blow-up system (84) for $\gamma = 0$

$$v_t = 0, \quad v = -\varphi_t, \quad w_s - v\varphi_s = 0$$

has an explicit solution

$$\varphi = s - tu_0(s, y), \quad v = u_0(s, y), \quad w = \partial_s^{-1} u_0 - \frac{u_0^2}{2} t,$$

from which it follows that $\partial_s \varphi$ has value zero "for the first time" at the time $\bar{T}_0$ (86) in the unique point $M_0 = (x_0, y_0, \bar{T}_0)$. This means with notation

$$\tilde{x}_0 = x_0 - \bar{T}_0 u_0(x_0, y_0), \quad \tilde{M}_0 = (\tilde{x}_0, y_0, \bar{T}_0),$$

that u_x has a blow-up in the point $\tilde{M}_0$.

We denote the blow-up system (85) by

$$\mathcal{L}(\varphi, v, w) = 0, \text{where } \mathcal{L} = (\mathcal{E}, \mathcal{R}, \mathcal{A}) \tag{87}$$

which we have to solve in the domain D_b. Set the field $Z = -\partial_t - 2\gamma\varphi_y\partial_y$, and the notation

$$Q = -\gamma\partial_y^2, \quad \dot{z} = \dot{w} - v\dot{\varphi}.$$

The choice of $\dot{z}$ is natural and can be explained by linearization of the relation $u(\Phi) = w$ which gives $\dot{u}(\Phi) + u'(\Phi)\dot{\Phi} = \dot{w}$ from what

$$\dot{z} \equiv \dot{u}(\Phi) = \dot{w} - v\dot{\varphi}.$$

Here the "physical" objects are u and $\dot{u}(\Phi)$ but not w and $\dot{w}$ which depend on the change of variables Φ. We can say that the introduction of $\dot{z}$ cancels the arbitrariness in the choices of w and Φ. Then

$$\mathcal{A} = w_s - v\varphi_s = 0, \quad \mathcal{E} = Z\varphi - Q\varphi - v = 0,$$

$$\mathcal{R} = -Zv - vQ\varphi + Qw - \frac{\gamma}{y}(w_y - v\varphi_y) = 0,$$

and the linearized blow-up system has the form (we note $\mathcal{L}'_{\varphi,v,w}(\dot{\varphi}, \dot{v}, \dot{w}) = \mathcal{L}'(\dot{\varphi}, \dot{v}, \dot{w})$)

$$\mathcal{E}'(\dot{\varphi}, \dot{v}, \dot{w}) = \dot{f}, \quad \mathcal{E}'(\dot{\varphi}, \dot{v}, \dot{w}) = -\dot{\varphi}_t - \dot{v} - 2\gamma\varphi_y\dot{\varphi}_y = Z\dot{\varphi} - \dot{v},$$

$$\mathcal{R}'(\dot{\varphi}, \dot{v}, \dot{w}) = \dot{g}, \; \mathcal{R}'(\dot{\varphi}, \dot{v}, \dot{w}) = -Z\dot{v} + Q\dot{z} + (Qv)\dot{\varphi} - (Q\varphi)\dot{v} - \frac{\gamma}{y}(\dot{z}_y - \dot{v}\varphi_y + \dot{\varphi}v_y),$$

$$\mathcal{A}'(\dot{\varphi}, \dot{v}, \dot{w}) = \dot{h}, \quad \mathcal{A}'(\dot{\varphi}, \dot{v}, \dot{w}) = \dot{z}_s + v_s\dot{\varphi} - \varphi_s\dot{v},$$

or simply

$$\mathcal{L}'(\dot{\varphi}, \dot{v}, \dot{w}) = \dot{l}. \tag{88}$$

Following the structure of [2, p.23] we find

$$Z\partial_s\dot{z} - \varphi_s Q\dot{z} + ((Q\varphi)\partial_s + \frac{\gamma}{y}\varphi_s\partial_y)\dot{z} + \alpha_1 Z\dot{\varphi} + \alpha_2\dot{\varphi} =$$
$$= -\varphi_s\mathcal{R}' + (Z + Q\varphi)\mathcal{A}' - (Z\varphi_s + \frac{\gamma}{y}\varphi_s v_y)\mathcal{E}', \tag{89}$$

$$Z^2\dot{\varphi} - (Q_1\varphi)Z\dot{\varphi} + (Q_1 v)\dot{\varphi} + Q_1\dot{z} = (Z - Q_1\varphi)\mathcal{E}' - \mathcal{R}'. \tag{90}$$

Here

$$\alpha_1 = -\partial_s\mathcal{E} - \frac{\gamma}{y}\varphi_s v_y, \quad \alpha_2 = -Q_1\mathcal{A} - \partial_s\mathcal{R}, \quad Q_1 = -Q + \frac{\gamma}{y}\partial_y.$$

The coefficients α_1 and α_2 are small if $\mathcal{E}$, $\mathcal{R}$, $\mathcal{A}$ and their derivatives are small. So the system (89), (90) in $\dot{\varphi}$, $\dot{z}$ is almost decoupled. In a Nash-Moser scheme aimed at solving $\mathcal{E} = 0$, $\mathcal{R} = 0$, $\mathcal{A} = 0$ we could view these terms with the coefficients α_i as "quadratic errors". But we cannot just neglect them, because this would correspond to solving the linearized system up to quadratic errors divided by φ_s, which is not acceptable in the framework of smooth functions.

We need to introduce the identities (89), (90) with the help of which we will solve our linearized blow-up system.

The idea (see for example [2]) is the following. Suppose that we can solve exactly in D_b the system (89), (90) in $(\dot{z}, \dot{\varphi})$ with $\mathcal{E}'$, $\mathcal{R}'$ and $\mathcal{A}'$ replaced by given quantities $\dot{f}$, $\dot{g}$ and $\dot{h}$. Determine now $\dot{v}$ from $\mathcal{E}' = \dot{f}$ using the relation $\mathcal{E}' = Z\dot{\varphi} - \dot{v}$. For the functions $(\dot{\varphi}, \dot{v}, \dot{w})$ thus obtained, we have then

$$\mathcal{E}' = \dot{f},\ \mathcal{R}' = \dot{g},\ (Z + \alpha_3)(\mathcal{A}' - \dot{h}) = 0.$$

Taking into account the boundary conditions on (φ, v, w) (hence on $(\dot{\varphi}, \dot{v}, \dot{w})$), suppose we can ensure that $\mathcal{A}' - \dot{h}$ vanishes on some part Γ of the boundary of D_b; suppose also that D_b is under the influence of Γ for Z; then we obtain $\mathcal{A}' = \dot{h}$, and the linearized blow-up system is exactly solved.

But before to do it, let us to pass from the free boundary domain to a fix one where we will solve our linearized blow-up system.

Consider the surface Σ through $\{t = 0, s = K\}$ which is characteristic for the operator $Z\partial_s - \varphi_s Q$

$$t = -\psi(s, y),$$

where ψ is solution of the Cauchy problem

$$(1 + \gamma(2\varphi_y)\psi_y)\psi_s + \gamma\varphi_s\psi_y^2 = 0, \quad \psi(K, y) = 0. \tag{91}$$

Equation (91) has, for small γ, a smooth solution in the appropriate domain. This solution is $O(\gamma)$ and decreases in s.

We now perform the change of variables

$$\tilde{x} = s, \quad \tilde{y} = y, \quad \tilde{t} = (\chi(\frac{t}{\eta}) - 1)t - (t + \psi)\chi(\frac{t}{\eta}), \tag{92}$$

where $\chi \in C^\infty$ is 0 near 1 and 1 near 0, and $\eta > 0$ is small enough. The still unknown domain

$$D_\psi = \{-A_0 \leq s \leq K, y \in [r_0, r_1], -\psi \leq t \leq \bar{t}_\gamma\}$$

is taken by this change into

$$\tilde{D} = \{-A_0 \leq \tilde{x} \leq K, \tilde{y} \in [r_0, r_1], -\bar{T} = -\bar{t}_\gamma \leq \tilde{t} \leq 0\}.$$

The change of variables (92) gives

$$\partial_s = \partial_{\tilde{x}} + \tilde{t}_s \partial_{\tilde{t}}, \quad \partial_y = \partial_{\tilde{y}} + \tilde{t}_y \partial_{\tilde{t}}, \quad \partial_t = \tilde{t}_t \partial_{\tilde{t}},$$

where

$$\tilde{t}_s = O(\gamma), \quad \tilde{t}_y = O(\gamma), \quad \tilde{t}_t = -1 + O(\gamma)$$

are known functions.

In the domain $\tilde{D}$ the blow-up time $\bar{T}$ is unknown and is the part of the problem, so we still have a free boundary problem. To handle this problem, we introduce a parameter λ close to zero and perform the change of variables

$$\tilde{x} = X, \quad \tilde{y} = Y, \quad \tilde{t} = \tilde{t}(T, \lambda) \equiv T + \lambda T(1 - \chi_1(T)), \tag{93}$$

where χ_1 is one near zero and zero near $-\bar{T}_0$ (defined in (86)).

This implies that

$$D_{bf} = \{(X, Y, T) | X \in [-A_0, K], \quad Y \in [r_0, r_1], \quad r_0 > 0, \quad T \in] - \bar{T}_0, 0]\}. \tag{94}$$

The two successive changes of variables (92), (93)

$$(s, y, t) \to (\tilde{x}, \tilde{y}, \tilde{t}) \to (X, Y, T)$$

imply that our blow-up system (87) are transformed into

$$\widetilde{\mathcal{L}}(\lambda, \varphi, v, w) = 0. \tag{95}$$

The system (89), (90) is also changed to some system $\widetilde{(89)}$, $\widetilde{(90)}$, the exact view of which we will find later.

We say that φ satisfies condition **(H)** in D_{bf} if, for some boundary point $M = (X, Y, -\bar{T}_0) \in \overline{D}_{bf}$

$$\begin{cases} \varphi_X \geq 0, \quad \varphi_X(X, Y, T) = 0 \Leftrightarrow (X, Y, T) = M, \\ \varphi_{XT}(M) < 0, \quad \nabla_{X,Y}(\varphi_X)(M) = 0, \nabla^2_{X,Y}(\varphi_X)(M) >> 0. \end{cases} \tag{96}$$

The problem we want to solve in D_{bf} is

1. $\widetilde{\mathcal{L}}(\lambda, \varphi, v, w) = 0,$
2. φ satisfies **(H)** in D_{bf}.

We are following the plan which is explained in details later:

1. We are assuming that we can solve the linearized system $\widetilde{(89)}$, $\widetilde{(90)}$ and so $\partial_{\varphi,v,w}\widetilde{\mathcal{L}}\dot{\Psi} = \dot{f}$ in flat functions with a tame estimate.

2. Resolution of $\widetilde{\mathcal{L}} = 0$ using the above fact by Nash-Moser iteration process reproducing in each step $\varphi^{(n)}$ satisfying the condition **(H)** ($\forall n$) with the help of some techniques based on the structure of **(H)** and the implicit function theorem (fundamental lemma of Alinhac).

3. We prove the point 1 for the system $\widetilde{(89)}$, $\widetilde{(90)}$.

For the start point of the Nash-Moser iteration process we choose

$$\lambda^{(0)} = 0,\ \varphi^{(0)} = \bar{\varphi}^{(0)},\ v^{(0)} = \bar{v}^{(0)},\ w^{(0)} = \bar{w}^{(0)}.$$

Let us now determinate the functions denoted by $\bar{\varphi}^{(0)}$, $\bar{v}^{(0)}$, $\bar{w}^{(0)}$.

For $\gamma = 0$ and $\lambda = 0$ the exact solution of blow-up system with initial conditions

$$\varphi(X, Y, 0) = X, \quad \partial_T\varphi(X, Y, 0) = u_0(X, Y)$$

is

$$\bar{\varphi}_0 = X + Tu_0(X, Y), \quad \bar{v}_0 = u_0(X, Y), \quad \bar{w}_0 = \partial_s^{-1}u_0 + \frac{1}{2}Tu_0^2,$$

we can also notice collecting all change of variables that

$$\bar{\varphi}_0(s, y, T) = \bar{\varphi}_0(s, y, \tilde{t}(T, 0)) = \bar{\varphi}_0(X, Y, T).$$

So for $\lambda = 0$ ($T = \tilde{t}$) the approximate solution of the first step of Nash-Moser process existing for the time $T \in]-\bar{T}_0, 0]$ can be obtained by gluing together the local true solution $(\bar{\varphi}, \bar{v}, \bar{w})$ of (95), which exists in a small strip $\{-\eta_1 \le T \le 0\}$ of D_{bf}, to $(\bar{\varphi}_0, \bar{v}_0, \bar{w}_0)$ in the following form:

$$\bar{\varphi}^{(0)}(X, Y, T) = \chi\left(\frac{-T}{\eta_1}\right)\bar{\varphi}(X, Y, T) + \left(1 - \chi\left(\frac{-T}{\eta_1}\right)\right)\bar{\varphi}_0(X, Y, T).$$

We have then [5, 6] that for this approximate solution

$$\widetilde{\mathcal{L}}(0, \bar{\varphi}^{(0)}, \bar{v}^{(0)}, \bar{w}^{(0)}) = \bar{l}^{(0)},$$

where $\bar{l}^{(0)}$ is smooth, flat on $\{X = M\}$, zero near $\{T = 0\}$, and zero for $\gamma = 0$.

Since the solution we start from has already all the good traces on $\{X = M\}$ and $\{T = 0\}$, we need only to solve the linearized system in flat functions.

The approximate solution $\bar{\varphi}^{(0)}$ satisfies, thanks to (75) and $\partial_X\bar{\varphi} > 0$ close to $\{T = 0\}$, the condition **(H)** (96) in point $M = (m_0, -\bar{T}_0)$, where $\bar{T}_0$ is from (86).

Solving $\widetilde{\mathcal{L}} = 0$ by a Nash-Moser iteration process, which we start from the point $(0, \bar{\varphi}^{(0)}, \bar{v}^{(0)}, \bar{w}^{(0)})$, further modifications of $\bar{\varphi}^{(0)}$ will yield functions not satisfying **(H)** anymore. So we have to make sure that we can reproduce at each step the new function φ satisfying the condition **(H)**. This is realized thanks to the “fundamental lemma” from [6].

For the multidimensional case when $n=3$ and $y=(y^1,y^2)$ we use [5, p.16] to determine the final form of the fixed domain D_{bf} where we want to solve our blow-up system (95), and so the domain D_{bf} is the domain bounded by the planes $X=-A_0$, $X=M$, $T=-\bar{T}_0$, $T=0$, the plane containing $(\delta_1, I_1 I_3)$, and the plane containing $(\delta_2, I_2 I_4)$. These planes have normal $n_\pm = (-\eta_1, \pm\nu, 1)$ and are described by

$$\delta_1 = \left\{T=0, Y-(y_0-y_1) = -\frac{\eta_1}{\nu}(X-M)\right\},\ I_1 = (y_0 - y_1 - \bar{T}_0/\nu, -\bar{T}_0),$$

$$\delta_2 = \left\{T=0, Y-(y_0-y_1) = \frac{\eta_1}{\nu}(X-M)\right\},\ I_2 = (y_0 + y_1 + \bar{T}_0/\nu, -\bar{T}_0),$$

$$I_3 = (Y = y_0 - y_1, T=0), \quad I_4 = (y_0 + y_1, T=0),$$

where y_0 is from (75), y_1 and ν are fixed such that $y_0 - y_1 \le y \le y_0 + y_1$, $0 < \bar{T}_0 < \frac{1}{2}\nu y_1$ (for the explication of the details see [5, p.16]). It is understood that A_0 and the small η_1 are chosen such that $\bar{\varphi}^{(0)}$ satisfies **(H)** for a point M interior to the lower boundary of D_{bf}.

The linearized operator of $\tilde{\mathcal{L}}$ at the point (λ, φ, v, w) is denoted by

$$\tilde{\mathcal{L}}'_{\lambda,\varphi,v,w}(\dot\lambda, \dot\varphi, \dot v, \dot w) = \partial_\lambda \tilde{\mathcal{L}}(\lambda,\varphi,v,w)\dot\lambda + \partial_\varphi \tilde{\mathcal{L}}(\lambda,\varphi,v,w)\dot\varphi + \partial_v \tilde{\mathcal{L}}(\lambda,\varphi,v,w)\dot v + \partial_w \tilde{\mathcal{L}}(\lambda,\varphi,v,w)\dot w.$$

We can see that with $q = \partial_\lambda \tilde{t} / \partial_T \tilde{t}$

$$\partial_\lambda \tilde{\mathcal{L}} + \partial_\varphi \tilde{\mathcal{L}}(\varphi_T q) + \partial_v \tilde{\mathcal{L}}(v_T q) + \partial_w \tilde{\mathcal{L}}(w_T q) = q \tilde{\mathcal{L}}_T.$$

Thus the linearized system

$$\tilde{\mathcal{L}}'_{(\lambda,\varphi,v,w)}(\dot\lambda, \dot\varphi, \dot v, \dot w) = \dot{l},$$

is equivalent to

$$\widetilde{\mathcal{L}'}_{\varphi,v,w}(\dot\Phi, \dot V, \dot W) = \dot{l} - q\dot\lambda \tilde{\mathcal{L}}_T \tag{97}$$

with

$$\dot\Phi = \dot\varphi - \dot\lambda q \varphi_T, \quad \dot V = \dot v - \dot\lambda q v_T, \quad \dot W = \dot w - \dot\lambda q w_T, \quad \dot Z = \dot W - v\dot\Phi,$$

here $\widetilde{\mathcal{L}'} = (\widetilde{\mathcal{E}'}, \widetilde{\mathcal{R}'}, \widetilde{\mathcal{A}'})$ denotes the linear system obtained from the linearized blow-up system (88) in the original variables s, y, t by the two successive changes of variables (92), (93).

Assume now that, at some stage of the Nash-Moser iteration process aimed at solving $\tilde{L} = 0$, the function φ satisfies **(H)**. We solve first (97), neglecting $q\dot\lambda\tilde{\mathcal{L}}_T$ in the right-hand side, since it is a quadratic error. We choose then, once $\dot\phi$ is known, $\dot\lambda$ such that

$$\varphi + \dot\varphi = \varphi + \dot\Phi + \dot\lambda q \varphi_T$$

satisfies again condition **(H)** for some point on the lower boundary of the fixed domain D_{bf}. It is possible using Alinhac's Fundamental lemma which can be found with the iteration scheme of following resolution of the problem in [6, p.110-112].

Hence, to finish our proof it is enough to solve the transformed linear system

$$\widetilde{\mathcal{E}'} = \dot{f},\ \widetilde{\mathcal{R}'} = \dot{g},\ \widetilde{\mathcal{A}'} = \dot{h}$$

in D_{fb}.

For this we use the system (89), (90) transformed by the two changes of variables into $\widetilde{(89)}$, $\widetilde{(90)}$ which we want to write now explicitly.

First let us introduce the following notations

$$\hat{Z} = \partial_T + \gamma z_0 \partial_Y, \quad S = \partial_X + \gamma s_0 \partial_T.$$

The composition of the two changes of variables operates the following transformation of operators (to avoid introducing unnecessary notation, we denote by $*$ known functions):

$$\partial_s = \partial_X + \gamma s_0(X, Y, T, \lambda)\partial_T \equiv S, \quad \partial_y = \partial_Y + \gamma * (X, Y, T, \lambda)\partial_T,$$

$$\partial_t = (-1 + \gamma * (X, Y, T, \lambda))(\partial_T \tilde{t})^{-1}\partial_T, \quad Z = (1 + \gamma * (X, Y, T, \lambda, \varphi, \varphi_Y, \varphi_T))\hat{Z}$$

and the transformed linearized system (89), (90) has the form

$$\hat{Z}S\dot{Z} + \gamma(S\varphi)N\dot{Z} + \gamma l_1(\dot{Z}) + \alpha_1 \hat{Z}\dot{\Phi} + \alpha_2 \dot{\Phi} = \dot{f}_1, \tag{98}$$

$$\hat{Z}^2\dot{\Phi} + \gamma\beta_1\hat{Z}\dot{\Phi} + \gamma\beta_2\dot{\Phi} + \gamma\hat{Z}H\dot{Z} + \gamma l_1'(\dot{Z}) = \dot{f}_2. \tag{99}$$

Here

1. $N = N_1\hat{Z}^2 + 2\gamma N_2 \hat{Z}\partial_Y + N_3\partial_Y^2$, with $N_1 = -O(\gamma)$, $N_3 = \partial_T\tilde{t} + O(\gamma)$,
2. $l_1(\dot{Z})$, $l_1'(\dot{Z})$ are linear combinations of $\nabla\dot{Z}$ and $\dot{Z}$, for example $l_1(\dot{Z}) = (N\varphi)S\dot{Z} + \frac{1}{Y}(S\varphi)\partial_Y\dot{Z}$,
3. H is a linear combination of $\hat{Z}$ and ∂_Y.

We can also denote $\gamma = \varepsilon$ because of its smallness.

Also introducing

$$\tilde{P} = \hat{Z}S\hat{Z} + \epsilon(S\varphi)N\hat{Z},$$

we set $\dot{Z} = \hat{Z}\dot{k}$ in the linearized system (98), (99).

So we have now to solve the system

$$\tilde{P}\dot{k} + \epsilon l_1(\hat{Z}\dot{k}) + \alpha_1\hat{Z}\dot{\Phi} + \alpha_2\dot{\Phi} = \dot{f}_1,$$
$$\hat{Z}^2\dot{\Phi} + \beta_1\hat{Z}\dot{\Phi} + \beta_2\dot{\Phi} + \epsilon\hat{Z}H\hat{Z}\dot{k} + \epsilon\beta_3\partial_Y^2\hat{Z}\dot{k} + \epsilon l_1'(\hat{Z}\dot{k}) = \dot{f}_2.$$

With the notations

$$A = S\varphi, \quad \delta = T - \bar{T}_0, \quad g = \exp\{h(x-t)\}, p^2 = \delta^\mu g, \quad |\cdot|_0 = |\cdot|_{L_2(D_{fb})},$$

we have the energy inequality (3.2.2) of [5, p.18] and the rest of the proof is absolutely identical to [5].

Having obtained the solution λ, φ, v, w of the blow-up system in the domain D_{bf} we construct as it is shown in [5, p.22-23] the solution u of (77) from which we go back to $\tilde{u}$ the KZK solution of (74) which will be periodic in x (thanks to the theorem 2 of the existing of the unique solution) if we take the initial data u_0 periodic in x, and $\tilde{u}_x$ has a blow-up at the point $(x_\gamma, y_\gamma, \bar{T}_\gamma)$.

3 Validity of the KZK approximation

The purpose of this section is to specify the theorem 1 and to show in which sense the KZK equation provides asymptotic solutions of the equation

$$\partial_t \rho + \nabla.(\rho u) = 0\,, \quad \rho(\partial_t u + (u.\nabla)u) = -\nabla p(\rho) + \epsilon\nu\Delta u\,.$$

The viscosity ν introduces some difference in the construction. We treat separately the cases $\nu = 0$ and $\nu \neq 0$.

For the case with no viscosity, $\nu = 0$, theorems of existence and of stability for the Euler system and for the KZK equation are valid for positive and negative but finite time.

In the viscous case we have global stability with sufficient small initial data for the KZK equation and large time (for positive time but under a smallness hypothesis of initial data up to infinity) existence and stability for Navier-Stokes system in a half space with a convenient condition on ρ.

However we start the viscous case with linear problem to compare the errors for the linear and non linear problems (see table 1).

3.1 Validity of the KZK approximation for non viscous thermoellastic media

On the one hand one considers the Euler system for $\tilde{\rho}_\epsilon(x_1, x', t)$, $\tilde{u}_\epsilon(x_1, x', t)$:

$$\partial_t \tilde{\rho}_\epsilon + \operatorname{div}(\tilde{\rho}_\epsilon \tilde{u}_\epsilon) = 0\,, \quad \tilde{\rho}_\epsilon[\partial_t \tilde{u}_\epsilon + (\tilde{u}_\epsilon.\nabla)\tilde{u}_\epsilon] = -\nabla p(\tilde{\rho}_\epsilon), \tag{100}$$

and on the other hand a non trivial solution I of the problem

$$c\partial^2_{\tau z} I - \frac{(\gamma+1)}{4\rho_0}\partial^2_\tau I^2 - \frac{c^2}{2}\Delta_y I = 0, \tag{101}$$

for some initial condition

$$I(\tau, 0, y) = I_0(\tau, y).$$

The solution I as a function of (τ, z, y) is periodic in τ of period L. One constructs in this case for the KZK approximation a solution for (x_1, t) positive and negative using initial data compact in x for $t = 0$.

The theorem 2 ensures for initial data $I(\tau, 0, y) \in H^{s'}$ with $s' > [\frac{n}{2}] + 1$ the existence of a solution $I(\tau, z, y) \in C(|z| < R; H^{s'}(\tau \times y))$ (for zero viscosity

$\nu = 0$). The existence of the smooth solution $\tilde{U}_\epsilon = (\tilde{\rho}_\epsilon, \tilde{u}_\epsilon)(t, x_1, x')$ $(0 \leq t \leq T)$ of Euler equation (100) is due to the theorem 5.1.1 from [14, p. 62].

With the notation

$$\tilde{\rho}_\epsilon = \rho_0 + \epsilon\rho_\epsilon \quad \tilde{u}_\epsilon = \epsilon u_\epsilon, \tag{102}$$

we take

$$p = p(\tilde{\rho}_\epsilon) = c^2\epsilon\rho_\epsilon + \frac{(\gamma-1)c^2}{2\rho_0}\epsilon^2{\rho_\epsilon}^2. \tag{103}$$

Then one constructs according to the formulas the functions

$$v(\tau, z, y) = \frac{c}{\rho_0} I(\tau, z, y), \tag{104}$$

$$w(\tau, z, y) = -\frac{c^2}{\rho_0}\left(\int_0^\tau \nabla_y I(s, z, y) ds + \int_0^L \frac{s}{L}\nabla_y I(s, z, y) ds\right), \tag{105}$$

$$\begin{aligned} v_1(\tau, z, y) = &-\frac{c^2}{\rho_0}\left(\int_0^\tau \partial_z I(s, z, y) ds + \int_0^L \frac{s}{L}\partial_z I(s, z, y) ds\right) + \\ &+\frac{(\gamma-1)}{2\rho_0^2} c I^2(\tau, z, y) - \frac{c(\gamma-1)}{2L\rho_0^2}\int_0^L I^2(\tau, z, y) d\tau. \end{aligned} \tag{106}$$

In the above formulas the terms containing $\int_0^L$ correspond to the definition of the operator ∂_τ^{-1}, which implies that all these functions are L-periodic in τ and of mean value 0.

Next introduce the densities and velocities

$$\overline{\rho}_\epsilon = \rho_0 + \epsilon I(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x'), \tag{107}$$

$$\overline{u}_{\epsilon,1} = \epsilon(v + \epsilon v_1)(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x'), \tag{108}$$

$$\overline{u}'_\epsilon = \epsilon^{\frac{3}{2}}\vec{w}(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x') \tag{109}$$

and eventually the expression:

$$\overline{U}_\epsilon = (\overline{\rho}_\epsilon, \overline{u}_\epsilon) = (\rho_0 + \epsilon I, \epsilon(v + \epsilon v_1, \sqrt{\epsilon}\vec{w}))(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x'). \tag{110}$$

We envisage the problem of approximation between the two systems: the exact system (100) and the approximate system obtained from a smooth solution of KZK equation (101):

$$\left\{ \begin{array}{l} \partial_t\overline{\rho}_\epsilon + \nabla_x.(\overline{\rho}_\epsilon\overline{u}_\epsilon) = \\ \epsilon^3\left(\rho_0\partial_z v_1 + \partial_z(Iv) - \frac{1}{c}\partial_\tau(Iv_1) + \nabla_y(Iw)\right) + \epsilon^4\partial_z(Iv_1) \end{array} \right. \tag{111}$$

$$
\begin{cases}
\bar{\rho}_\epsilon(\partial_t \bar{u}_{\epsilon,1} + \bar{u}_\epsilon \cdot \nabla . \bar{u}_{\epsilon,1}) + \partial_{x_1} p(\bar{\rho}_\epsilon) = \\
\epsilon^3 \left(I\partial_\tau v_1 - \frac{1}{2c} I \partial_\tau v^2 + \frac{\rho_0}{2}\partial_z v^2 - \frac{\rho_0}{c}\partial_\tau(vv_1) + \rho_0 w \nabla_y v + \frac{(\gamma-1)}{2\rho_0} c^2 \partial_z I^2 \right) + \\
\epsilon^4 \left(\frac{I}{2}\partial_z v^2 - \frac{1}{c} I \partial_\tau (vv_1) + I w \nabla_y v + \rho_0 \partial_z (vv_1) - \frac{\rho_0}{2c}\partial_\tau v_1^2 + \rho_0 w \nabla_y v_1 \right) + \\
\epsilon^5 \left(I\partial_z(vv_1) - \frac{1}{2c} I \partial_\tau v_1^2 + I w \nabla_y v_1 + \frac{\rho_0}{2}\partial_z v_1^2 \right)
\end{cases}
\tag{112}
$$

$$
\begin{cases}
\bar{\rho}_\epsilon(\partial_t \bar{u}'_\epsilon + \bar{u}_\epsilon \cdot \nabla . \bar{u}'_\epsilon) + \partial_{x'} p(\bar{\rho}_\epsilon) = \epsilon^{\frac{5}{2}} \left(\frac{\gamma-1}{2\rho_0} c^2 \nabla_y I^2 \right) + \\
\epsilon^{\frac{7}{2}} \left(\rho_0 v \partial_z w - \frac{\rho_0}{c} v_1 \partial_\tau w + \frac{\rho_0}{2} \nabla_y w^2 - \frac{I}{c} v \partial_\tau w \right) + \\
\epsilon^{\frac{9}{2}} \left(\rho_0 v_1 \partial_z w + I v \partial_z w - \frac{I}{c} v_1 \partial_\tau w + \frac{I}{2} \nabla_y w^2 \right) + \epsilon^{\frac{11}{2}} I v_1 \partial_z w
\end{cases}
\tag{113}
$$

The system (111)-(113) could be written in the form

$$
\begin{aligned}
&\partial_t \overline{\rho}_\epsilon + \nabla.(\overline{\rho u}_\epsilon) = R_1 \, , \\
&\overline{\rho}_\epsilon(\partial_t \overline{u}_\epsilon + \overline{u}_\epsilon \cdot \nabla . \overline{u}_\epsilon) + \nabla p(\overline{\rho}_\epsilon) = \vec{R_2},
\end{aligned}
$$

with notation R_1 for the rest of (111), and $\vec{R_2}$ for the rest of (112) and of (113).

To ensure that R_1, $\vec{R_2} \in L_\infty((-R,R);L_2)$ we need that $\partial_z^2 I \in L_\infty((-R,R);L_2)$ and choose in theorem 2 $s > \max\{4, [\frac{n}{2}]+1\}$.

The existence of the smooth "true" solution of Euler equation (100) $\widetilde{U}_\epsilon = (\tilde{\rho}_\epsilon, \tilde{u}_\epsilon)(t, x_1, x')$ $(0 \le t < T_0)$ with $\nabla.\widetilde{U}_\epsilon(\cdot, t) \in C^0([0,T_0); H^{s-1})$ for $s-1 > \frac{n}{2}$, with the same initial data $\widetilde{U}_\epsilon|_{t=0} = \overline{U}_\epsilon|_{t=0}$ is still due to the theorem 5.1.1 from [14, p. 62].

Remark 12 *As soon as the time of the existence of Euler system solution T_0 is finite, the interval $[0, T_0)$ is maximal (see [14, p. 62]) in the sense that*

$$
\lim_{t \uparrow T_0} \sup \|\nabla.\widetilde{U}(\cdot, t)\|_{L_\infty} = \infty.
$$

To precise the order of the blow-up time T_0 we can observe it in the simplified model of Euler equation, particulary the Burgers equation with, as in our case, the initial conditions of order ϵ:

$$
\partial_t u + u \frac{\partial u}{\partial x} = 0, \quad u|_{t=0} = \epsilon u_0.
$$

As it is well known, the first derivative

$$
\nabla_x u = \frac{\epsilon \nabla_x u_0}{1 - \epsilon \nabla_x u_0 t}
$$

exists for time

$$t < \frac{1}{\epsilon |\nabla_x u_0|}.$$

Resulting to our problem, we will consider the solution of Euler equation for the time $t \in [0, \frac{T}{\epsilon})$*, where* T *is a constant and* ϵ *is small.*

So the solution given by the KZK approximation and a true solution of Euler system with the same data at time $t = 0$ can be compared according to the following theorem

Theorem 7 *Suppose that there exists the solution* I *of the KZK equation (for some initial data from* H^s*) such that*

- $I(\tau, z, y)$ *is L-periodic with respect to* τ *and defined for* $|z| \leq R$ *and* $y \in \mathcal{R}^{n-1}$*,*
- *assume that*

  $$z \mapsto I(\tau, z, y) \in C(]-R, R[; H^{s'}(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}_y^{n-1})) \cap C^1(]-R, R[; H^{s'-2}(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}_y^{n-1}))$$

  for $s > \max\{4, [\frac{n}{2}] + 1\}$.

(the existence of such solution is proved in theorem 2).

Let $\overline{U}_\epsilon$ *be the approximate solution of the isentropic Euler equation deduced from a solution of the KZK equation with the help of (107)-(109), (104)-(106). Then the function* $\overline{U}_\epsilon(x_1, x', t) = \overline{U}_\epsilon(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon} x')$ *given by the formula (110) is defined in*

$$\mathcal{R}_t \times (\Omega_\epsilon = \{|x_1| < \frac{R}{\epsilon} - ct\} \times \mathcal{R}_{x'}^{n-1})$$

and is smooth enough according to the above procedure.

Consider now the solution of the Euler system (100) in a cone (see the figure 3.1)

$$C(t) = \{0 < s < t\} \times Q_\epsilon(s) = \{x = (x_1, x') : |x_1| \leq \frac{R}{\epsilon} - Ms, \ M \geq c, \ x' \in \mathcal{R}^{n-1}\}$$

with the initial data

$$(\bar{\rho}_\epsilon - \rho_\epsilon)|_{t=0} = 0, \quad (\bar{u}_\epsilon - u_\epsilon)|_{t=0} = 0.$$

Then (see [14, p. 62]) there exists T_0 *such that for the time interval* $0 \leq t \leq \frac{T_0}{\epsilon}$ *there exists the classical solution* $U_\epsilon = (\rho_\epsilon, u_\epsilon)$ *of the Euler system (100) in a cone*

$$C(T) = \{0 < t < T | T < \frac{T_0}{\epsilon}\} \times Q_\epsilon(t) \tag{114}$$

with

$$\|\nabla . U_\epsilon\|_{L_\infty([0, \frac{T_0}{\epsilon}[; H^{s-1})} < \epsilon C \ \ \text{for} \ \ s > [\frac{n}{2}] + 1.$$

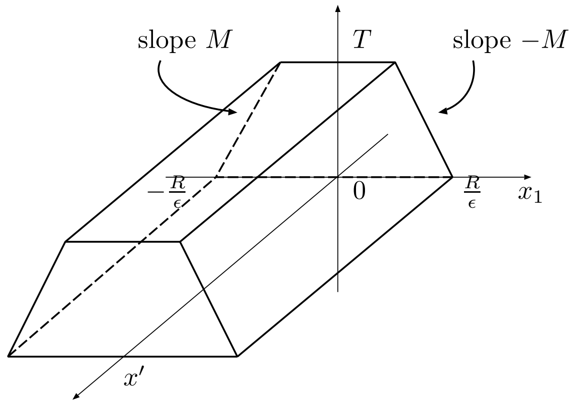

Figure 1: The cone $C(T)$.

Then there exists a constant C such that for any ϵ small enough the solutions $\widetilde{U}_\epsilon \overset{note}{=} (\rho_\epsilon, \rho_\epsilon u_\epsilon)$ and $\bar{U}_\epsilon \overset{note}{=} (\overline{\rho}_\epsilon, \overline{\rho}_\epsilon \overline{u}_\epsilon)$, which have been determinate as above in the cone (114) with the same initial data (133) satisfy the estimate

$$\|\bar{U}_\epsilon - \widetilde{U}_\epsilon\|^2_{L_2(Q_\epsilon(t))} \leq \epsilon^5 e^{C\|\nabla.\bar{U}_\epsilon\|_{L_\infty(C(T))}t} \leq \epsilon^5 e^{C\epsilon t}. \tag{115}$$

Remark 13 *As soon as the booth solutions $\overline{U}_\epsilon$ and $\widetilde{U}_\epsilon$ are in $C([0, \frac{T_0}{\epsilon}[; H^s)$ in the cone for any $s > \max\{4, [\frac{n}{2}] + 1\}$, we can apply the operator $\Lambda^{s'}$ with $s' = s - 4$ and obtain the same estimate (115) but for the norm $\|\cdot\|_{H^{s'}(Q_\epsilon(t))}$.*

Proof. We need to use here a technique due to Dafermos [14].

As it is known the isentropic Euler equation admits a convex entropy $\eta(\widetilde{U}_\epsilon)$, which is the function:

$$\eta(\widetilde{U}_\epsilon) = \rho_\epsilon h(\rho_\epsilon) + \rho_\epsilon \frac{|u_\epsilon|^2}{2} \text{ with } h'(\rho_\epsilon) = \frac{p(\rho_\epsilon)}{\rho_\epsilon^2}. \tag{116}$$

Having assumed $\widetilde{U}_\epsilon = (\rho_\epsilon, \rho_\epsilon u_\epsilon)^T$, we can rewrite the Euler system

$$\partial_t \widetilde{U}_\epsilon + \nabla.F(\widetilde{U}_\epsilon) = 0, \quad \text{where } F(\widetilde{U}_\epsilon) = (\rho_\epsilon u_\epsilon, \rho_\epsilon u_\epsilon^2 + p(\rho_\epsilon))^T$$

in terms of entropy (116):

$$\partial_t \eta(\widetilde{U}_\epsilon) + \nabla.q(\widetilde{U}_\epsilon) = 0, \quad \text{where } q(\widetilde{U}_\epsilon) = u_\epsilon(\eta(\widetilde{U}_\epsilon) + p(\rho_\epsilon)).$$

So we have two systems

$$\begin{cases} \partial_t \eta(\widetilde{U}_\epsilon) + \nabla.q(\widetilde{U}_\epsilon) = 0, \\ \partial_t \widetilde{U}_\epsilon + \nabla.F(\widetilde{U}_\epsilon) = 0, \end{cases}$$

$$\begin{cases} \partial_t \eta(\overline{U}_\epsilon) + \nabla . q(\overline{U}_\epsilon) = \frac{\eta(\overline{U}_\epsilon)+p(\bar{\rho}_\epsilon)}{\bar{\rho}_\epsilon} R_1 + \bar{u}_\epsilon \vec{R}_2, \\ \partial_t \overline{U}_\epsilon + \nabla . F(\overline{U}_\epsilon) = \vec{R}, \end{cases}$$

where $R_1 = O(\epsilon^3)$ is the rest of the first equation of Euler system, $\vec{R}_2 = (O(\epsilon^3), O(\epsilon^{\frac{5}{2}})) = O(\epsilon^{\frac{5}{2}})$ is the rest of the second equation of Euler system in two directions x_1 and x', and $\vec{R} = (R_1, \vec{R}_2) = O(\epsilon^{\frac{5}{2}})$. So $\frac{\eta(\overline{U}_\epsilon)+p(\bar{\rho}_\epsilon)}{\bar{\rho}_\epsilon} R_1 + \bar{u}_\epsilon \vec{R}_2 = O(\epsilon^3)$.

Let us compute

$$\begin{aligned} &\frac{\partial}{\partial t}(\eta(\widetilde{U}_\epsilon) - \eta(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon)) = -\nabla.(q(\widetilde{U}_\epsilon) - q(\overline{U}_\epsilon)) - \\ &-\vec{R}^T \eta''(\overline{U})(\widetilde{U} - \overline{U}) - \frac{\eta(\overline{U}_\epsilon) + p(\bar{\rho}_\epsilon)}{\bar{\rho}_\epsilon} R_1 - \bar{u}_\epsilon \vec{R}_2 + \\ &+\eta'(\overline{U}_\epsilon)\nabla.(F(\widetilde{U}_\epsilon) - F(\overline{U}_\epsilon)) - \eta'(\overline{U}_\epsilon)\vec{R} - (\partial_t \overline{U}_\epsilon)^T \eta''(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon), \end{aligned} \tag{117}$$

and using the property for convex entropy $\eta''(U)F'(U) = (F'(U))^T \eta''(U)$ the last term is

$$-(\partial_t \overline{U}_\epsilon)^T \eta''(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) = \nabla.\overline{U}_\epsilon^T (F'(\overline{U}_\epsilon))^T \eta''(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) = \nabla.\overline{U}_\epsilon^T \eta''(\overline{U}_\epsilon) F'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon).$$

Integrate (117) over the cone $C(t)$ (cf. (114)). The use of the Green formula gives:

$$\begin{aligned} &\int_{|x_1|<\frac{R}{\epsilon}-Mt} \int_{\mathcal{R}^{n-1}} (\eta(\widetilde{U}_\epsilon) - \eta(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon)(x,t)dx \\ &- \int_{|x_1|<\frac{R}{\epsilon}} \int_{\mathcal{R}^{n-1}} (\eta(\widetilde{U}_\epsilon) - \eta(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon)(x,0)dx \\ &= -\int_{\partial C(t)} (n_t(\eta(\widetilde{U}_\epsilon) - \eta(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon)d\sigma \\ &- \int_{\partial C(t)} n_{x_1}(q(\widetilde{U}_\epsilon) - q(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)(F(\widetilde{U}_\epsilon) - F(\overline{U}_\epsilon)))d\sigma \\ &- \int_{C(t)} \nabla.\overline{U}_\epsilon^T \eta''(\overline{U}_\epsilon)(F(\widetilde{U}_\epsilon) - F(\overline{U}_\epsilon) - F'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon))dxds - \\ &- \int_{C(t)} (\frac{\eta(\overline{U}_\epsilon) + p(\bar{\rho}_\epsilon)}{\bar{\rho}_\epsilon} R_1 + \bar{u}_\epsilon \vec{R}_2 + \eta'(\overline{U}_\epsilon)\vec{R} - \vec{R}^T \eta''(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon))dxds \end{aligned} \tag{118}$$

With the help of the facts that the entropy η is convex

$$\eta(\widetilde{U}_\epsilon) - \eta(\overline{U}_\epsilon) - \eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon - \overline{U}_\epsilon) \geq \alpha|\widetilde{U}_\epsilon - \overline{U}_\epsilon|^2,$$

and associated entropy flux q is related with η by relation [14, p. 52]

$$q'(U) = \eta'(U)F'(U),$$

we obtain that

$$q(\widetilde{U}_\epsilon)-q(\overline{U}_\epsilon)=q'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon-\overline{U}_\epsilon)+O(|\widetilde{U}_\epsilon-\overline{U}_\epsilon|^2)=\eta'(\overline{U}_\epsilon)F'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon-\overline{U}_\epsilon)+O(|\widetilde{U}_\epsilon-\overline{U}_\epsilon|^2),$$

so, using the Taylor expansion

$$F(\widetilde{U}_\epsilon)-F(\overline{U}_\epsilon)=F'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon-\overline{U}_\epsilon)+O(|\widetilde{U}_\epsilon-\overline{U}_\epsilon|^2),$$

$$F(\widetilde{U}_\epsilon)-F(\overline{U}_\epsilon)-F'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon-\overline{U}_\epsilon)\leq C|\widetilde{U}_\epsilon-\overline{U}_\epsilon|^2,$$

$$q(\widetilde{U}_\epsilon)-q(\overline{U}_\epsilon)-\eta'(\overline{U}_\epsilon)(F(\widetilde{U}_\epsilon)-F(\overline{U}_\epsilon))\leq C|\widetilde{U}_\epsilon-\overline{U}_\epsilon|^2.$$

At last one can always choose our cone with the help of a constant $M\geq c$ such that $\alpha n_t+Cn_{x_1}>0$ and

$$-\int_{\partial C(t)}(\alpha n_t+Cn_x)\|\widetilde{U}_\epsilon-\overline{U}_\epsilon\|^2_{L_2(Q_\epsilon(s))}d\sigma<0.$$

Taking the same initial data $\overline{U}_\epsilon|_{t=0}=\widetilde{U}_\epsilon|_{t=0}$, we obtain

$$\|\widetilde{U}_\epsilon-\overline{U}_\epsilon\|^2_{L_2(Q_\epsilon(t))}\leq C\|\nabla.\overline{U}_\epsilon\|_{L_\infty(C(T))}\int_0^t\|\widetilde{U}_\epsilon-\overline{U}_\epsilon\|^2_{L_2(Q_\epsilon(s))}ds+Kt\epsilon^5.$$

Here the constants K and C do not depend on t.

Therefore applying the Gronwall lemma one has

$$\|\widetilde{U}_\epsilon-\overline{U}_\epsilon\|^2_{L_2(Q_\epsilon(t))}\leq K\epsilon^5\int_0^t e^{C(t-s)\|\nabla.\overline{U}_\epsilon\|_{L_\infty(C(T))}}ds. \tag{119}$$

As soon as the difference of the solutions has the order $\widetilde{U}_\epsilon-\overline{U}_\epsilon=O(\epsilon)$, also $\nabla.\overline{U}_\epsilon=O(\epsilon)$, and the left side of (119) has the order $O(\epsilon^2)$, so we have that in the cone $C(T)$ our estimate always remains as ϵ^2

$$\epsilon^5\int_0^t e^{C\epsilon(t-s)}ds<\epsilon^2.$$

3.2 Validity of the KZK approximation for viscous thermoellastic media

One has seen in theorem 2 that the solution $u(x,y,t)$ of the KZK equation with the term of viscosity $\beta>0$ or $\nu>0$ defines globally in time $t>0$ for rather small initial data. The solution $I(\tau,z,y)$ of the KZK equation is the asymptotic form of Navier-Stokes system. We note that $I(\tau,z,y)=I(t-\frac{x_1}{c},\epsilon x_1,\sqrt{\epsilon}x')$ is defined for $x_1>0$ (as soon as z becomes the time variable according to the KZK derivation from section 1). For this reason the approximate domain of validity of the KZK approximation for viscous thermoellastic media is the half space $x_1>0$, $t>0$, $x'\in R^{n-1}$.

3.2.1 Linearized KZK equation

From the Navier-Stokes system

$$\partial_t\widetilde{\rho}+\operatorname{div}(\widetilde{\rho}\widetilde{u})=0\,,\quad \widetilde{\rho}(\partial_t\widetilde{u}+(\widetilde{u}\cdot\nabla)\widetilde{u})=-\nabla p(\widetilde{\rho})+\epsilon\nu\Delta\widetilde{u} \tag{120}$$

which is well posed for rather small initial data in half space $\{x_1>0, x'\in\mathcal{R}^{n-1}, t>0\}$, the isentropic linear Navier-Stokes system can be obtained with the help of the choice

$$\widetilde{\rho}=\rho_0+\epsilon^2\rho,\quad \widetilde{u}=\epsilon^2 u,\quad p=p(\widetilde{\rho})=c^2\epsilon^2\rho$$

not taking into account the terms of order $O(\epsilon^3)$:

$$\partial_t\rho+\rho_0\nabla.u=0, \tag{121}$$
$$\rho_0\partial_t u+\nabla p(\rho)=\epsilon\nu\Delta u, \tag{122}$$

which have a unique global solution.

Combining ∂_t and ∇ applied to (121) and (122) we obtain two decoupled linear equations

$$\partial_t^2\rho-c^2\triangle\rho=\epsilon\frac{\nu}{\rho_0}\triangle\partial_t\rho, \tag{123}$$
$$\partial_t^2 u-c^2\nabla\operatorname{div}u=\epsilon\frac{\nu}{\rho_0}\triangle\partial_t u. \tag{124}$$

The existence of the smooth solution u of (124) follows from the lemma.

Lemma 4 *The equation (124) has unique regular global on time solution in the half space $\{x_1>0,t>0\}$, with regular initial $u|_{t=0}=u_0$, $u_t|_{t=0}=u_0$ and boundary $u|_{x_1=0}=u_b$ conditions (u_b has the same properties as the initial condition for the KZK equation $I(\tau,0,y)$, and u_0 as $I(-\frac{x_1}{c},\epsilon x_1,\sqrt{\epsilon}x')$).*

Proof. If $u_b=0$, it follows from the relation

$$\frac{d}{dt}\left(\int_{x_1>0}|\partial_t u|^2dx+c^2\int_{x_1>0}|\nabla.u|^2dx\right)+\frac{\epsilon\nu}{\rho_0}\int_{x_1>0}|\nabla.\partial_t u|^2ds-$$
$$-c^2\int_{x_1=0}\nabla.u\partial_t u dx'-\frac{\epsilon\nu}{\rho_0}\int_{x_1=0}\nabla.\partial_t u\partial_t u dx'=0,$$

which gives in this case

$$\frac{d}{dt}\int_{x_1>0}\left(|\partial_t u|^2+c^2|\nabla.u|^2\right)dx\le 0.$$

If we take $u_b\neq 0$, then we can write that $u=v+\varphi$, such that $v|_{t=0}$, $v_t|_{t=0}$ are the same as for u and zero in the boundary, and φ is a function of a compact support with zero initial conditions and the same boundary condition as u. We can construct such φ using the trace theorem, thanks to the regularity of u_b.

Let us put $u = v + \varphi$ in (124)

$$\partial_t^2(v+\varphi) - c^2\nabla\operatorname{div}(v+\varphi) = \epsilon\frac{\nu}{\rho_0}\triangle\partial_t(v+\varphi).$$

From where we have

$$\partial_t^2 v - c^2\nabla\operatorname{div} v - \epsilon\frac{\nu}{\rho_0}\triangle\partial_t v = g(\varphi).$$

We multiply now the equation on $\partial_t v$ and integrate on x:

$$\frac{d}{dt}\left(\int_{x_1>0}|\partial_t v|^2 dx + c^2\int_{x_1>0}|\nabla . v|^2 dx\right) + \frac{\epsilon\nu}{\rho_0}\int_{x_1>0}|\nabla .\partial_t v|^2 ds \leq$$
$$\leq \frac{1}{2}\int_{x_1>0}|g(\varphi)|^2 dx + \frac{1}{2}\int_{x_1>0}|\partial_t v|^2 dx,$$

applying now the Gronwall lemma for the equality type $y' \leq a + y$ we obtain the global existence in time of the solution. $\square$

Having the solution u of (124) with boundary condition $u|_{x_1=0} = u^1$ we can find ρ from (121) according to the formula

$$\rho(x,t) = \rho(x,0) - \rho_0\int_0^t \nabla . u(x,s)ds. \tag{125}$$

This ρ satisfies (123) with boundary condition

$$\partial_t\rho|_{x_1=0} = -\rho_0\nabla . u|_{x_1=0}.$$

And so we can envisage instead of the system (121), (122) the following system

$$\partial_t\rho + \rho_0\nabla . u = 0, \tag{126}$$
$$\partial_t^2 u - c^2\nabla\operatorname{div} u = \epsilon\frac{\nu}{\rho_0}\triangle\partial_t u. \tag{127}$$

If we pass to the variables (τ, z, y) in (121), we obtain the linear part of KZK for $\rho(t, x_1, x') = I(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x')$

$$\partial_t^2\rho - c^2\triangle\rho - \epsilon\frac{\nu}{\rho_0}\triangle\partial_t\rho = \epsilon(2c\partial_{\tau z}^2 I - c^2\triangle_y I - \frac{\nu}{\rho_0 c^2}\partial_\tau^3 I) +$$

$$+\epsilon^2(-c^2\partial_z^2 I + 2\frac{\nu}{\rho_0 c}\partial_\tau^2\partial_z I - \frac{\nu}{\rho_0}\triangle_y\partial_\tau I) - \epsilon^3\frac{\nu}{\rho_0}\partial_z^2\partial_\tau I.$$

Suppose that I is the smooth periodic solution of linear KZK equation

$$2c\partial_{\tau z}^2 I - c^2\triangle_y I - \frac{\nu}{\rho_0 c^2}\partial_\tau^3 I = 0, \tag{128}$$

then

$$\partial_t^2\bar\rho-c^2\triangle\bar\rho-\epsilon\triangle\partial_t\bar\rho=\epsilon^2(-c^2\partial_z^2 I+2\frac{\nu}{\rho_0 c}\partial_\tau^2\partial_z I-\frac{\nu}{\rho_0}\triangle_y\partial_\tau I)-\epsilon^3\frac{\nu}{\rho_0}\partial_z^2\partial_\tau I. \quad (129)$$

And the equation of the same form holds for $\bar u=(v,\sqrt{\epsilon}\vec w)$

$$\partial_t^2\bar u-c^2\nabla\operatorname{div}\bar u-\epsilon\triangle\partial_t\bar u=\begin{cases}\epsilon^2(-c^2\partial_z^2 v+2\frac{\nu}{c}\partial_\tau^2\partial_z v-\nu\triangle_y\partial_\tau v)-\epsilon^3\nu\partial_z^2\partial_\tau v,\\ \\ \epsilon^{\frac52}(2\frac{\nu}{c}\partial_\tau^2\partial_z\vec w-\nu\triangle_y\partial_\tau\vec w)-\epsilon^{\frac72}\nu\partial_z^2\partial_\tau\vec w,\end{cases} \quad (130)$$

where the functions v and $\vec w$ are constructed according to the formulas (104), (105). So for the exact system (126), (127) the approximate system has the form

$$\partial_t\bar\rho+\rho_0\nabla.\bar u=\epsilon\rho_0(\partial_z v+\nabla_y\vec w), \quad (131)$$

$$\partial_t^2\bar u-c^2\nabla\operatorname{div}\bar u-\epsilon\frac{\nu}{\rho_0}\triangle\partial_t\bar u=\begin{cases}\epsilon^2(-c^2\partial_z^2 v+2\frac{\nu}{c}\partial_\tau^2\partial_z v-\nu\triangle_y\partial_\tau v)-\epsilon^3\nu\partial_z^2\partial_\tau v,\\ \\ \epsilon^{\frac52}(2\frac{\nu}{c}\partial_\tau^2\partial_z\vec w-\nu\triangle_y\partial_\tau\vec w)-\epsilon^{\frac72}\nu\partial_z^2\partial_\tau\vec w,\end{cases} \quad (132)$$

In this case we easily obtain what follows

Theorem 8 *Let $I(\tau,z,y)$ be a solution of the linear KZK equation (128) L-periodic and mean value zero with respect to τ and defined in the half space $z>0,\tau\in\mathcal{R}/L\mathcal{Z},y\in\mathcal{R}^{n-1}$, decays for $z\to\infty$. Assume that*

$$z\mapsto I(\tau,z,y)\in C([0,\infty[;H^{s'}(\mathcal{R}/L\mathcal{Z}\times\mathcal{R}_y^{n-1}))\cap C^1([0,\infty[;H^{s'-2}(\mathcal{R}/L\mathcal{Z}\times\mathcal{R}_y^{n-1}))$$

for $s>\max\{6,[\frac{n}{2}]+1\}$.

Let $\overline{U}_\epsilon=(\bar\rho,\bar u)$ be the smooth solution of the approximate system (131), (132) deduced from a solution of the KZK equation with the help of (104)-(105). Then the function $\overline{U}_\epsilon(x_1,x',t)=\overline{U}_\epsilon(x_1-ct,\epsilon x_1,\sqrt{\epsilon}x')$ given by the formula

$$\overline{U}_\epsilon(x_1,x',t)=(I,(v,\sqrt{\epsilon}\vec w))(x_1-ct,\epsilon x_1,\sqrt{\epsilon}x')$$

is defined in the half space

$$\{x_1>0,\ x'\in\mathcal{R}^{n-1},\ t>0\}$$

and is smooth enough according to the above procedure.

If $U=(\rho,u)$ is the solution of (126), (127) in $\{x_1>0,\ x'\in\mathcal{R}^{n-1},\ t>0\}$ with the same rather small initial data and boundary condition

$$\begin{aligned}&(\bar u-u)|_{t=0}=\partial_t(\bar u-u)|_{t=0}=0,\\&(\bar u-u)|_{x_1=0}=0,\quad \partial_t\rho|_{x_1=0}=-\rho_0\nabla.u|_{x_1=0},\end{aligned} \quad (133)$$

then there exists a constant $C>0$ such that the following estimates hold

$$\int_{x_1>0}|\partial_t(\bar u_\epsilon-u)|^2+|\nabla.(\bar u_\epsilon-u)|^2dx_1dx'\le C\epsilon^4t^2, \quad (134)$$

which remains smaller than the order ϵ^2 for the time $0 < t < \frac{T}{\epsilon^2}$, and

$$\int_{x_1>0} |\bar{\rho}_\epsilon - \rho|^2 dx_1 dx' \leq \epsilon^2 e^{Ct}, \tag{135}$$

which remains smaller than the order ϵ^2 for the time $0 < t < T \ln \frac{1}{\epsilon}$.

The approximation result is true for the solution $\widetilde{U}$ of the linearized system (121), (122) since $\widetilde{U} \equiv U$ (with $\partial_t \rho|_{x_1=0} = -\rho_0 \nabla.u|_{x_1=0}$).

Proof.

For the difference $\bar{u} - u$ we have

$$\partial_t^2(\bar{u} - u) - c^2 \triangle(\bar{u} - u) = \epsilon \triangle \partial_t(\bar{u} - u) + \epsilon^2 \vec{R},$$

where

$$\vec{R} = \begin{cases} \epsilon^2(-c^2\partial_z^2 v + 2\frac{\nu}{c}\partial_\tau^2\partial_z v - \nu\triangle_y\partial_\tau v) - \epsilon^3\nu\partial_z^2\partial_\tau v, \\ \\ \epsilon^{\frac{5}{2}}(2\frac{\nu}{c}\partial_\tau^2\partial_z\vec{w} - \nu\triangle_y\partial_\tau\vec{w}) - \epsilon^{\frac{7}{2}}\nu\partial_z^2\partial_\tau\vec{w}, \end{cases}$$

the rest of (129) bounded, under the smoothness hypotheses of the KZK solution I, at least in $L_\infty([0, \infty[, L_2)$.

Multiplying the last equation by $\partial_t(\bar{u} - u)$ one obtains

$$\frac{d}{dt}\int_{x_1>0}(|\partial_t(\bar{u}-u)|^2 + c^2|\nabla.(\bar{u}-u)|^2)dx_1 dx' - c^2\int_{x_1=0}\partial_t(\bar{u}-u)\cdot\nabla.(\bar{u}-u)dx' =$$

$$= \epsilon\int_{x_1=0}\partial_t(\bar{u}-u)\nabla.\partial_t(\bar{u}-u)dx' - \epsilon\int_{x_1>0}|\nabla.\partial_t(\bar{u}-u)|^2 dx_1 dx' + \epsilon^2\int_{x_1>0}R\partial_t(\bar{u}-u)dx_1 dx'.$$

In the same time

$$\int_{x_1=0}\partial_t(\bar{u} - u)\nabla.\partial_t(\bar{u} - u)dx' + c^2\int_{x_1=0}\partial_t(\bar{u} - u)\cdot\nabla.(\bar{u} - u)dx' = 0,$$

because one can choose $\bar{u}|_{x_1=0} = u|_{x_1=0}$(cf. (133)). From where

$$\frac{d}{dt}\int_{x_1>0}(|\partial_t(\bar{u}-u)|^2 + c^2|\nabla.(\bar{u}-u)|^2)dx_1 dx' \leq \epsilon^2 C\left(\int_{x_1>0}|\partial_t(\bar{u} - u)|^2 + c^2|\nabla.(\bar{u} - u)|^2 dx_1 dx'\right)^{\frac{1}{2}}.$$

The above estimate has the form

$$\frac{d}{dt}(\sqrt{E})^2 \leq C\epsilon^2\sqrt{E} \quad \Rightarrow \quad \frac{d}{dt}\sqrt{E} \leq C\epsilon^2$$

and then

$$\frac{d}{dt}\left(\int_{x_1>0}|\partial_t(\bar{u} - u)|^2 + c^2|\nabla.(\bar{u} - u)|^2 dx_1 dx'\right)^{\frac{1}{2}} \leq C\epsilon^2,$$

which gives the estimate (134).

In our construction the expression $\sqrt{E} = O(1)$ because $\bar{u} = O(1)$, $u = O(1)$, $(\bar{u} - u) = O(1)$, which ensures that the estimate remains smaller the order ϵ^2 for the time $0 < t < \frac{T}{\epsilon^2}$.

Passing now to $(\bar{\rho} - \rho)$ we have the relation

$$\partial_t(\bar{\rho} - \rho) + \rho_0 \nabla.(\bar{u} - u) = \epsilon \rho_0 (\partial_z v + \nabla_y \vec{w}).$$

We multiply the equation by $(\bar{\rho} - \rho)$ in $L_2(\{x_1 > 0\})$

$$\frac{d}{dt} \int_{x_1>0} |\bar{\rho} - \rho|^2 dx + \rho_0 \int_{x_1>0} (\bar{\rho} - \rho) \nabla.(\bar{u} - u) dx = \epsilon \rho_0 \int_{x_1>0} \vec{R}(\bar{\rho} - \rho) dx,$$

and using the estimate (134) for the time $t < \frac{T}{\epsilon^2}$, we obtain the estimate

$$\|\bar{\rho} - \rho\|_{L_2(\{x_1>0\})} \leq \epsilon C e^{Ct},$$

which remains smaller the order ϵ^2 for the time $t < T \ln \frac{1}{\epsilon}$. $\square$

3.2.2 The general nonlinear case

On the one hand one considers the system:

$$\partial_t \rho_\epsilon + \mathrm{div}(\rho_\epsilon u_\epsilon) = 0\,, \quad \rho_\epsilon[\partial_t u_\epsilon + (u_\epsilon \cdot \nabla) u_\epsilon] = -\nabla p(\rho_\epsilon) + \epsilon \nu \Delta u_\epsilon \tag{136}$$

and on the other hand a non trivial solution I of the problem

$$c\partial^2_{\tau z} I - \frac{(\gamma+1)}{4\rho_0} \partial^2_\tau I^2 - \frac{\nu}{2c^2 \rho_0} \partial^3_\tau I - \frac{c^2}{2} \Delta_y I = 0, \tag{137}$$

for some initial data

$$I(\tau, 0, y) = I_0(\tau, y).$$

The solution I as a function of (τ, z, y) is periodic in τ of period L. The theorem 2 implies for any initial data I_0 $\mathcal{R}^{\mathrm{per}}_\tau \times \Omega_y$ with small enough H^s ($s > [\frac{n}{2}] + 1$) norm (with respect to ν) there exists a unique solution which decays for $z \to \infty$.

We still envisage our problem in the half space $x_1 > 0$, $t > 0$ with the assumption that $u \to 0, \rho \to \rho_0$ for $|x| \to \infty$.

Let us construct the approximate system to the Navier-Stokes system (136).

We take as the state equation

$$p = p(\rho_\epsilon) = c^2 \epsilon \tilde{\rho}_\epsilon + \frac{(\gamma-1)c^2}{2\rho_0} \epsilon^2 \tilde{\rho}^2_\epsilon.$$

Then one constructs according to the formulas the functions: v from (104), w from (105) and

$$v_1(\tau, z, y) = -\frac{c^2}{\rho_0} \left(\int_0^\tau \partial_z I(s, z, y) ds + \int_0^L \frac{s}{L} \partial_z I(s, z, y) ds \right) +$$

$$+ \frac{(\gamma-1)}{2\rho_0^2} c I^2 - \frac{c(\gamma-1)}{2L\rho_0^2} \int_0^L I^2(\tau, z, y) d\tau + \frac{\nu}{c\rho_0^2} \partial_\tau I. \tag{138}$$

In the above formula the terms containing $\int_0^L$ correspond to the definition of the operator ∂_τ^{-1}, which implies that v_1 is L-periodic in τ and of mean value 0. To exclude the derivative on z from (138) we find from the KZK equation that

$$-\frac{c^2}{\rho_0}\partial_\tau^{-1}\partial_z I = -\frac{(\gamma+1)c}{4\rho_0^2}I^2 + \frac{c(\gamma+1)}{4\rho_0^2 L}\int_0^L I^2 ds + \frac{\nu}{2c\rho^2}\partial_\tau I + \frac{c^3}{2\rho_0}\partial_\tau^{-2}\triangle_y I,$$

and so

$$v_1(\tau,z,y) = \frac{c^3}{2\rho_0}\partial_\tau^{-2}\triangle_y I(\tau,z,y) + \frac{(\gamma-1)}{4\rho_0^2}cI^2(\tau,z,y) -$$
$$-\frac{c(\gamma-1)}{4L\rho_0^2}\int_0^L I^2(\tau,z,y)d\tau + \frac{3\nu}{2c\rho_0^2}\partial_\tau I(\tau,z,y). \tag{139}$$

Next we introduce the densities and velocities (107)-(109) and construct the function $\overline{U}$ (110).

In particular for $t=0$ one has functions defined for $x_1>0$ because I was well defined for any $z>0$

$$\overline{\rho}_\epsilon(0,x_1,x') = \rho_0 + \epsilon I(-\frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x'),$$
$$\overline{u}_\epsilon\mid_{t=0} = (\overline{u}_{\epsilon,1}, \overline{u}'_\epsilon)(-\frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon}x')$$

and for $x_1=0$ one has L-periodic functions of mean value zero

$$\overline{\rho}_\epsilon(t,0,x') = \rho_0 + \epsilon I(t,0,\sqrt{\epsilon}x'), \tag{140}$$
$$\overline{u}_\epsilon(t,0,x') = (\overline{u}_{\epsilon,1}, \overline{u}'_\epsilon)(t,0,\sqrt{\epsilon}x'). \tag{141}$$

Since for (136) in our case on the boundary $u|_{x_1=0} = \epsilon\tilde{u}_\epsilon|_{x_1=0}$ is small and so $|u|_{x_1=0}| < c$, we have only two cases in our boundary: a subsonic inflow boundary when the first velocity component is positive $u_1|_{x_1=0} > 0$, and a subsonic outflow boundary when the first component of velocity is negative $u_1|_{x_1=0} < 0$.

We also notice that, thanks to (139),

$$\bar{u}_1|_{x_1=0} = \left(\epsilon\frac{c}{\rho_0}I + \epsilon^2 G(I)\right)(t,0,\sqrt{\epsilon}x') = \left.\left(\epsilon\frac{c}{\rho_0}I + \epsilon^2 G(I)\right)\right|_{z=0} =$$
$$= \epsilon\frac{c}{\rho_0}I_0(t,y) + \epsilon^2 G(I_0)(t,y),$$

with

$$G(I) = \frac{c^3}{2\rho_0}\partial_\tau^{-2}\triangle_y I + \frac{(\gamma-1)}{4\rho_0^2}cI^2 - \frac{c(\gamma-1)}{4L\rho_0^2}\int_0^L I^2 d\tau + \frac{3\nu}{2c\rho_0^2}\partial_\tau I, \tag{142}$$

so the boundary conditions for $\bar{u}_1$ are defined by the initial conditions for KZK equation and are L-periodic on t and of mean value zero. Therefore, the sign of

$\bar{u}_1|_{x_1=0}$ is the same as the sign of I_0 (because the term $G(I_0)$ has higher order of smallness on ϵ).

The function $\overline{U}_\epsilon \overset{note}{=} (\bar{\rho}_\epsilon, \bar{u}_\epsilon)$, defined in (110) by densities and velocities from (107)-(109), is solution of the problem

$$\begin{cases} \partial_t \bar{\rho}_\epsilon + \nabla.(\bar{\rho}_\epsilon \bar{u}_\epsilon) = \\ \epsilon^3 \left(\rho_0 \partial_z v_1 + \partial_z (Iv) - \frac{1}{c}\partial_\tau (Iv_1) + \nabla_y (Iw) \right) + \epsilon^4 \partial_z (Iv_1), \end{cases} \tag{143}$$

$$\begin{cases} \bar{\rho}_\epsilon (\partial_t \bar{u}_{\epsilon,1} + (\bar{u}_\epsilon \cdot \nabla) \bar{u}_{\epsilon,1}) + \partial_{x_1} p(\bar{\rho}_\epsilon) - \epsilon\nu\Delta \bar{u}_{\epsilon,1} = \\ \epsilon^3 \left(I\partial_\tau v_1 - \frac{1}{2c} I \partial_\tau v^2 + \frac{\rho_0}{2}\partial_z v^2 - \frac{\rho_0}{c}\partial_\tau (v v_1) + \rho_0 w \nabla_y v + \right. \\ \left. + \frac{(\gamma-1)}{2\rho_0} c^2 \partial_z I^2 + \frac{2\nu}{c}\partial^2_{z\tau} v - \nu \Delta_y v - \frac{\nu}{c^2}\partial^2_\tau v_1 \right) + \\ \epsilon^4 \left(\frac{I}{2}\partial_z v^2 - \frac{1}{c} I \partial_\tau (v v_1) + I w \nabla_y v + \rho_0 \partial_z (v v_1) - \frac{\rho_0}{2c}\partial_\tau v_1^2 + \right. \\ \left. + \rho_0 w \nabla_y v_1 - \nu \partial_z^2 v + \frac{2\nu}{c}\partial^2_{z\tau} v_1 - \nu \Delta_y v_1 \right) + \\ \epsilon^5 \left(I \partial_z (v v_1) - \frac{1}{2c} I \partial_\tau v_1^2 + I w \nabla_y v_1 + \frac{\rho_0}{2}\partial_z v_1^2 - \nu \partial_z^2 v_1 \right) \end{cases} \tag{144}$$

$$\begin{cases} \bar{\rho}_\epsilon (\partial_t \bar{u}'_\epsilon + (u_\epsilon \cdot \nabla) \bar{u}'_\epsilon) + \partial_{x'} p(\bar{\rho}_\epsilon) - \epsilon\nu\Delta \bar{u}'_\epsilon = \\ \epsilon^{\frac{5}{2}} \left(\frac{\gamma-1}{2\rho_0} c^2 \nabla_y I^2 + \frac{\nu}{\rho_0}\Delta_y I \right) + \\ \epsilon^{\frac{7}{2}} \left(\rho_0 v \partial_z w - \frac{\rho_0}{c} v_1 \partial_\tau w + \frac{\rho_0}{2}\nabla_y w^2 - \frac{I}{c} v \partial_\tau w - \nu \Delta w + \frac{2\nu}{c}\partial^2_{z\tau} w \right) + \\ \epsilon^{\frac{9}{2}} \left(\rho_0 v_1 \partial_z w + I v \partial_z w - \frac{I}{c} v_1 \partial_\tau w + \frac{I}{2}\nabla_y w^2 - \nu \partial_z^2 w \right) + \\ \epsilon^{\frac{11}{2}} \left(I v_1 \partial_z w \right) \end{cases} \tag{145}$$

Here to control the terms in right sides we need that $\partial_z^3 I \in L_\infty([0,\infty[, L_2(\tau \times y))$, so in theorem 2 we take $s > \max\{6, [\frac{n}{2}] + 1\}$. Then the rest of (143)-(145) is bounded in L_2.

So we have two systems: the system (136) and

$$\partial_t \bar{\rho}_\epsilon + \nabla.(\bar{\rho}_\epsilon \bar{u}_\epsilon) = \epsilon^{\frac{5}{2}} R_1, \quad \bar{\rho}_\epsilon [\partial_t \bar{u}_\epsilon + (\bar{u}_\epsilon \cdot \nabla)\bar{u}_\epsilon] + \nabla p(\bar{\rho}_\epsilon) - \epsilon\nu\Delta\bar{u}_\epsilon = \epsilon^{\frac{5}{2}} \vec{R}_2, \tag{146}$$

where $\epsilon^{\frac{5}{2}} R_1 = O(\epsilon^3)$ is the rest of the first equation of Navier-Stokes system (143), $\epsilon^{\frac{5}{2}} \vec{R}_2 = (O(\epsilon^3), O(\epsilon^{\frac{5}{2}})) = O(\epsilon^{\frac{5}{2}})$ is the rest of the second equation of Navier-Stokes system (144), (145) in two directions x_1 and x'.

Remark 14 *As we have the term of viscosity $\epsilon\nu\triangle u$, where ϵ is fixed rather small parameter, ν is a constant, and in our case u is of the order ϵ, so then the phenomenon of boundary layer is exclude.*

Theorem 9 *Suppose that the initial data of the KZK Cauchy problem $I_0(t,y) = I_0(t,\sqrt{\epsilon}x')$ is such that*

1. *it is periodic on t with the period L and of mean value zero,*
2. *for fixed t it has the same sign for all $y \in \mathcal{R}^{n-1}$, and for $t \in]0,L[$ change the sign, i.e., $I_0 = 0$, only finite number times,*
3. *$I_0(t,y) \in H^{s'}(\{t \geq 0\} \times \mathcal{R}^{n-1})$ for $s' > \max\{6, [\frac{n}{2}]+1\}$,*
4. *I_0 is sufficiently small in the sense of the theorem 2 and $I_0 = \epsilon^p \tilde{I}_0$, $p \geq 0$.*

Then there exits a unique global in time solution $I(\tau,z,y)$ of (137) for $z = \epsilon x_1 > 0$. Therefore the functions $\bar{\rho}_\epsilon$, $\bar{u}_\epsilon = (\bar{u}_{\epsilon,1}, \bar{u}'_\epsilon)$, defined by (107)-(109) in the half space

$$\{x_1 > 0,\ x' \in \mathcal{R}^{n-1},\ t > 0\}, \tag{147}$$

are smooth:

$$\bar{\rho}_\epsilon \in C([0,\infty[, H^{s'}(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}^{n-1}_y)) \cap C^1([0,\infty[; H^{s'-2}(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}^{n-1}_y)), \tag{148}$$

$$\bar{u}_\epsilon \in C([0,\infty[; H^{s'-2}(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}^{n-1}_y)) \cap C^1([0,\infty[; H^{s'-4}(\mathcal{R}/L\mathcal{Z} \times \mathcal{R}^{n-1}_y)). \tag{149}$$

The Navier-Stokes system (136) in the half space with the initial data (133) and following boundary conditions

$$(\bar{u}_\epsilon - u_\epsilon)|_{x_1=0} = 0,$$

and when the first component of the velocity is positive $u_{\epsilon,1}|_{x_1=0} > 0$ (i.e. at points where the fluid enters the domain) the additional boundary condition

$$(\bar{\rho}_\epsilon - \rho_\epsilon)|_{x_1=0} = 0.$$

When $u_{\epsilon,1}|_{x_1=0} \leq 0$ there is not any boundary condition for ρ_ϵ.

Suppose also that $u_\epsilon \to 0$, $\rho_\epsilon \to \rho_0$ as $|x| \to \infty$.

Then there exists a constant $T_0 > 0$ such that for all $t < \frac{T_0}{\epsilon^{2+p}}$ there exists a unique solution $U_\epsilon = (\rho_\epsilon, u_\epsilon)$ of Navier-Stokes system (136) with the same smoothness as (148), (149).

Remark 15 *Since the boundary conditions for the Navier-Stokes system are periodic and of mean value zero on t, the first component of the velocity $u_1|_{x_1=0}$ changes the sign and the inflow part of the boundary goes after the inflow one and so on. On the variables x' we have the constant sign of $u_1|_{x_1=0}$. This hypothesis follows from the physical reason of works of Zabolotskaya (see [10]). In [10] one takes as the initial conditions for the KZK equation (which correspond to the boundary condition for u_1) the expression*

$$I(\tau,0,y) = -F(y)\sin\tau.$$

The amplitude distribution $F(y)$ is taken two types:

- *for a Gaussian beam*

$$F(y) = e^{-y^2},$$

- *for a beam with a polynomial amplitude*

$$F(y) = \begin{cases} (1-y^2)^2, & y \le 1, \\ 0, & y > 1. \end{cases}$$

Proof. Using the fact of the convex entropy for the isentropic Euler equation $\eta(\widetilde{U}_\epsilon)$, which is the function (see (116)):

$$\eta(\widetilde{U}_\epsilon) = \rho_\epsilon h(\rho_\epsilon) + \rho_\epsilon \frac{|u_\epsilon|^2}{2} = H(\rho_\epsilon) + \frac{1}{\rho_\epsilon}\frac{m^2}{2} \text{ with } h'(\rho_\epsilon) = \frac{p(\rho_\epsilon)}{\rho_\epsilon^2},\ u_\epsilon = \frac{m}{\rho_\epsilon},$$

and their first and second derivatives

$$\eta'(\widetilde{U}_\epsilon) = \begin{bmatrix} H'(\rho_\epsilon) - \frac{1}{\rho_\epsilon^2}\frac{m^2}{2} \\ \frac{m}{\rho_\epsilon} \end{bmatrix}^T = \begin{bmatrix} H'(\rho_\epsilon) - \frac{u_\epsilon^2}{2} \\ u_\epsilon \end{bmatrix}^T, \tag{150}$$

$$\eta''(\widetilde{U}_\epsilon) = \begin{bmatrix} H''(\rho_\epsilon) + \frac{m^2}{\rho_\epsilon^3} & -\frac{m}{\rho_\epsilon^2} \\ -\frac{m}{\rho_\epsilon^2} & \frac{1}{\rho_\epsilon} \end{bmatrix} = \begin{bmatrix} H''(\rho_\epsilon) + \frac{u_\epsilon^2}{\rho_\epsilon} & -\frac{u_\epsilon}{\rho_\epsilon} \\ -\frac{u_\epsilon}{\rho_\epsilon} & \frac{1}{\rho_\epsilon} \end{bmatrix}. \tag{151}$$

Have assumed $\widetilde{U}_\epsilon = (\rho_\epsilon, \rho_\epsilon u_\epsilon)^T = (\rho_\epsilon, m)^T$, we can rewrite the Navier-Stokes system

$$\partial_t \widetilde{U}_\epsilon + \nabla . F(\widetilde{U}_\epsilon) - \epsilon\nu \begin{bmatrix} 0 \\ \triangle u_\epsilon \end{bmatrix} = 0, \quad \text{where } F(\widetilde{U}_\epsilon) = \begin{bmatrix} \rho_\epsilon u_\epsilon \\ \rho_\epsilon u_\epsilon^2 + p(\rho_\epsilon) \end{bmatrix}$$

in terms of entropy (116):

$$\partial_t \eta(\widetilde{U}_\epsilon) + \nabla . q(\widetilde{U}_\epsilon) - \epsilon\nu \begin{bmatrix} 0 \\ u_\epsilon \triangle u_\epsilon \end{bmatrix} = 0, \quad \text{where } q(\widetilde{U}) = u_\epsilon(\eta(\widetilde{U}_\epsilon) + p(\rho_\epsilon)).$$

Entropy estimate for isentropic Navier-Stokes equation on the half space and existence result

Let us multiply the Navier-Stokes system, from the left, by $2\widetilde{U}_\epsilon^T \eta''(U_\epsilon)$

$$2\widetilde{U}_\epsilon^T \eta''(\widetilde{U}_\epsilon)\partial_t \widetilde{U}_\epsilon + 2\widetilde{U}_\epsilon^T \eta''(\widetilde{U}_\epsilon) F'(\widetilde{U}_\epsilon)\nabla . \widetilde{U}_\epsilon - \epsilon\nu 2\widetilde{U}_\epsilon^T \eta''(\widetilde{U}_\epsilon) \begin{bmatrix} 0 \\ \triangle u_\epsilon \end{bmatrix} = 0.$$

Note that

$$\widetilde{U}_\epsilon^T \eta''(\widetilde{U}_\epsilon) \begin{bmatrix} 0 \\ \triangle u_\epsilon \end{bmatrix} = 0,$$

and that

$$2\widetilde{U}_\epsilon^T \eta''(\widetilde{U}_\epsilon)\partial_t \widetilde{U}_\epsilon = \partial_t[\widetilde{U}_\epsilon^T \eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon] - 2\widetilde{U}_\epsilon^T \partial_t \eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon.$$

Moreover, by virtue of $\eta''(U)F'(U) = (F'(U))^T\eta''(U)$,

$$2\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)\nabla.\widetilde{U}_\epsilon = \nabla.[\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon] - 2\widetilde{U}_\epsilon^T\nabla.[\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)]\widetilde{U}_\epsilon.$$

Integrating by $[0,t]\times\{x_1>0\}$ $(x'\in\mathcal{R}^{n-1})$ we have

$$\int_0^t\int_{x_1>0}\partial_t[\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon]dxds + \int_0^t\int_{x_1>0}\nabla.[\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon]dxds-$$

$$-2\int_0^t\int_{x_1>0}\widetilde{U}_\epsilon^T\partial_t\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon dxds - 2\int_0^t\int_{x_1>0}\widetilde{U}_\epsilon^T\nabla.[\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)]\widetilde{U}_\epsilon dxds = 0.$$

One integrates now by parts

$$-2\int_0^t\int_{x_1>0}\widetilde{U}_\epsilon^T\partial_t\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon dxds = -2\int_{x_1>0}\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon dx + 2\int_{x_1>0}\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon|_{t=0}dx+$$

$$+4\int_0^t\int_{x_1>0}\partial_t\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon dxds,$$

$$-2\int_0^t\int_{x_1>0}\widetilde{U}_\epsilon^T\nabla.[\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)]\widetilde{U}_\epsilon dxds = -2\int_0^t\int_{x_1=0}\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon dx'ds+$$

$$+4\int_0^t\int_{x_1>0}\nabla.\widetilde{U}_\epsilon^T[\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)]\widetilde{U}_\epsilon dxds,$$

noticing that

$$4\int_0^t\int_{x_1>0}\partial_t\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon dxds + \int_0^t\int_{x_1>0}\nabla.\widetilde{U}_\epsilon^T[\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)]\widetilde{U}_\epsilon dxds = 0,$$

we result in

$$\int_{x_1>0}\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon dx - \int_{x_1>0}\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon|_{t=0}dx - \int_0^t\int_{\mathcal{R}^{n-1}}\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)F'(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon|_{x_1=0}dx'ds = 0.$$

Recall that $\eta''(\widetilde{U}_\epsilon)$ is positive definite, so that

$$\widetilde{U}_\epsilon^T\eta''(\widetilde{U}_\epsilon)\widetilde{U}_\epsilon \geq \delta|\widetilde{U}_\epsilon|^2,$$

for some $\delta>0$.

Therefore, we obtain for the initial data

$$\widetilde{U}_0 = \left[\begin{array}{c}\rho_0+\epsilon I\\ \epsilon\left(\rho_0+\epsilon I\right)\left(\frac{c}{\rho_0}I+\epsilon v_1,\sqrt{\epsilon}\vec{w}\right)\end{array}\right]\left(-\frac{x_1}{c},\epsilon x_1,\sqrt{\epsilon}x'\right)$$

and for the first component of velocity the relation

$$\int_{x_1>0}\widetilde{U}_\epsilon^2dx - \int_{x_1>0}\widetilde{U}_0^2dx - \int_0^t\int_{\mathcal{R}^{n-1}}\widetilde{U}_{1,\epsilon}^T\eta''(\widetilde{U}_{1,\epsilon})F'(\widetilde{U}_{1,\epsilon})\widetilde{U}_{1,\epsilon}|_{x_1=0}dx'ds \leq 0.$$

Let us now consider the integral on the boundary. With notation $u_1 = u_{1,\epsilon}$ for the first component of velocity, we see that

$$\widetilde{U}_{1,\epsilon}^T \eta''(\widetilde{U}_{1,\epsilon}) F'(\widetilde{U}_{1,\epsilon}) \widetilde{U}_{1,\epsilon} = (\rho_\epsilon, \rho_\epsilon u_1) \begin{pmatrix} H''(\rho_\epsilon) + \frac{u_\epsilon^2}{2} & -\frac{u_1}{\rho_\epsilon} \\ -\frac{u_1}{\rho_\epsilon} & \frac{1}{\rho_\epsilon} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -u_\epsilon^2 + p'(\rho_\epsilon) & 2u_1 \end{pmatrix} \begin{pmatrix} \rho_\epsilon \\ \rho_\epsilon u_1 \end{pmatrix} =$$

$$= \rho_\epsilon^2 u_1 \left(H''(\rho_\epsilon) + \frac{u_\epsilon^2}{2} \right) - u_1 \left(\left(-u_\epsilon^2 + p'(\rho_\epsilon) \right) \rho_\epsilon + 2\rho_\epsilon u_1^2 \right) +$$

$$+ \rho_\epsilon u_1 \left(-u_1^2 + \frac{1}{\rho_\epsilon} \left(\left(-u_\epsilon^2 + p'(\rho_\epsilon) \right) \rho_\epsilon + 2\rho_\epsilon u_1^2 \right) \right) = u_1 \rho_\epsilon \left(p'(\rho_\epsilon) + \rho_\epsilon \frac{u_\epsilon^2}{2} \right) - u_1^3 \rho_\epsilon,$$

as soon as $H''(\rho) = \frac{p'(\rho)}{\rho}$.

Let us consider the initial condition $I_0(t,y)$ for the KZK equation of the type of the remark 15 and we suppose without loss of generality that $I_0 = 0$ for $t \in]0, L[$ only once in the point $\frac{L}{2}$, precisely we suppose that the sign of u_1 is changing in the following way:

- $u_1 \leq 0$ for $t \in [0 + (k-1)L, \frac{L}{2} + (k-1)L]$ $(k = 1, 2, 3, ...)$
- and $u_1 > 0$ for $t \in (\frac{L}{2} + (k-1)L, kL)$ $(k = 1, 2, 3, ...)$.

If $t \in [0, \frac{L}{2}]$ (for $k = 1$) the first component of velocity is negative

$$u_1|_{x_1=0} = \epsilon \frac{c}{\rho_0} I_0(t,y) + \epsilon^2 G(I_0)(t,y) < 0,$$

where $G(I_0)$ is L-periodic and of mean value zero from (142) ($u_1|_{x_1=0} = 0$ for $t = 0, \frac{L}{2}$), then

$$u_1 \rho_\epsilon \left(p'(\rho_\epsilon) + \rho_\epsilon \frac{u_\epsilon^2}{2} \right) - u_1^3 \rho_\epsilon < 0$$

since we have the negative term of order ϵ^2 and the positive term $-u_1^3 \rho_\epsilon$ of order ϵ^3. Therefore, for $t \in [0, \frac{L}{2}]$

$$\int_{x_1 > 0} \widetilde{U}_\epsilon^2 dx \leq \int_{x_1 > 0} \widetilde{U}_0^2 dx.$$

If $t \in (\frac{L}{2}, L)$ the first component of velocity is positive $u_1|_{x_1=0} > 0$, then we also impose $\rho_\epsilon|_{x_1=0} = \rho_0 + \epsilon I_0(t,y)$ where $I_0(t,y)$ is the initial condition for the KZK equation.Then we have that $u_1^3 \rho_\epsilon$ has the good sign, for the term

$$u_1 \rho_\epsilon \left(p'(\rho_\epsilon) + \rho_\epsilon \frac{u_\epsilon^2}{2} \right) > 0$$

we see that on the boundary it has the form

$$\epsilon \left(\frac{c}{\rho_0} I_0 + \epsilon v_1|_{z=0} \right) (\rho_0 + \epsilon I_0) \left(c^2 \epsilon + \frac{(\gamma-1)c^2}{\rho_0} \epsilon^2 I_0 + (\rho_0 + \epsilon I_0) \left(\left(\epsilon \frac{c}{\rho_0} I_0 + \epsilon^2 v_1|_{z=0} \right)^2 + \right.\right.$$
$$\left.\left. + \epsilon^3 \vec{w}^2|_{z=0} \right) \right) \leq \epsilon^2 C_0 I_0,$$

for some constant $C_0 = c^3 + \delta$ and so

$$\int_0^t \left(u_1 \rho_\epsilon \left(p'(\rho_\epsilon) + \rho_\epsilon \frac{u_\epsilon^2}{2} \right) - u_1^3 \rho_\epsilon \right) ds =$$

$$= \int_0^{\frac{L}{2}} \left(u_1 \rho_\epsilon \left(p'(\rho_\epsilon) + \rho_\epsilon \frac{u_\epsilon^2}{2} \right) - u_1^3 \rho_\epsilon \right) ds + \int_{\frac{L}{2}}^t \left(u_1 \rho_\epsilon \left(p'(\rho_\epsilon) + \rho_\epsilon \frac{u_\epsilon^2}{2} \right) - u_1^3 \rho_\epsilon \right) ds \leq$$

$$\leq - \left| \int_0^{\frac{L}{2}} \left(u_1 \rho_\epsilon \left(p'(\rho_\epsilon) + \rho_\epsilon \frac{u_\epsilon^2}{2} \right) - u_1^3 \rho_\epsilon \right) ds \right| + \int_{\frac{L}{2}}^L u_1 \rho_\epsilon \left(p'(\rho_\epsilon) + \rho_\epsilon \frac{u_\epsilon^2}{2} \right) ds - \int_{\frac{L}{2}}^L u_1^3 \rho_\epsilon dt \leq$$

$$\leq \epsilon^2 C_0 \int_{\frac{L}{2}}^L I_0 dt = \epsilon^{2+p} C_0 \int_{\frac{L}{2}}^L \tilde{I}_0 dt = \epsilon^{2+p} \widetilde{K},$$

where $\widetilde{K} = O(1)$ is a positive constant non depending on time and $p \geq 0$ is the order of "the sufficient small" initial data I_0.

Since $\widetilde{U}_0 = O(1)$ we obtain for all $t \leq L + \frac{L}{2}$ (for $t \in [L, L + \frac{L}{2}]$ $u_1|_{x_1=0} < 0$ and $\int_L^{L+\frac{L}{2}} \left(u_1 \rho_\epsilon \left(p'(\rho_\epsilon) + \rho_\epsilon \frac{u_\epsilon^2}{2} \right) - u_1^3 \rho_\epsilon \right) ds < 0$) the estimate

$$\int_{x_1>0} \widetilde{U}_\epsilon^2 dx \leq \int_{x_1>0} \widetilde{U}_0^2 dx + \epsilon^{2+p} \widetilde{K}.$$

But for $t < 2L + \frac{L}{2}$ we have, thanks to the periodicity of I_0,

$$\int_{x_1>0} \widetilde{U}_\epsilon^2 dx \leq \int_{x_1>0} \widetilde{U}_0^2 dx + 2\epsilon^{2+p} \widetilde{K}.$$

Then we conclude that for $t < \tau L + \frac{L}{2}$

$$\int_{x_1>0} \widetilde{U}_\epsilon^2 dx \leq \int_{x_1>0} \widetilde{U}_0^2 dx + \tau \epsilon^{2+p} \widetilde{K}.$$

To keep the sense of the bounded a priori estimate we need to impose that

$$\tau \epsilon^{2+p} \widetilde{K} = O(1), \quad \text{or} \quad \tau < \frac{1}{\widetilde{K} \epsilon^{2+p}}.$$

So for $t < L \left(\frac{1}{\widetilde{K} \epsilon^{2+p}} + \frac{1}{2} \right) = T$, or shortly $t < \frac{T_0}{\epsilon^{2+p}}$ with a constant $T_0 = O(1)$, we obtain that $\widetilde{U}_\epsilon \in L_\infty(]0, T[, L_2(\{x_1 > 0\} \times \mathcal{R}^{n-1}))$.

If $I_0 = 0$ for $t \in]0, L[$ finite number times, m times, beginning for example by negative sign, then we have the a priori estimate for $t < L\tau = T$, where $\tau < \frac{T_0}{(\widetilde{K}_1 + \ldots + \widetilde{K}_r)\epsilon^{2+p}}$, $r = [\frac{m+1}{2}]$.

To prove now that $\widetilde{U}_\epsilon \in L_\infty(]0, T[, H^{s'-2}(\{x_1 > 0\} \times \mathcal{R}^{n-1}))$ with s' from the condition of the theorem and where $s' - 1$ corresponds to the regularity of the initial condition $\widetilde{U}_0$, we use the result of [18, p. 352] for incompletely parabolic problems. We also obtain that $\partial_t \widetilde{U}_\epsilon \in L_\infty(]0, T[, H^{s'-4}(\{x_1 > 0\} \times \mathcal{R}^{n-1}))$.

Using the standard Faedo-Galerkin method with the theorem about a sequential compactness of the unit ball in the Hilbert space we obtain the existence of the unique solution of Navier-Stokes system. More precisely $\rho_\epsilon \in C([0,T[, H^{s'}(\{x_1>0\}\times \mathcal{R}^{n-1}))\cap C^1([0,T[, H^{s'-2}(\{x_1>0\}\times \mathcal{R}^{n-1}))$ and $u_\epsilon \in C([0,T[, H^{s'-2}(\{x_1>0\}\times \mathcal{R}^{n-1}))\cap C^1([0,T[, H^{s'-4}(\{x_1>0\}\times \mathcal{R}^{n-1}))$. $\square$

Theorem 10 *Suppose the assumptions of the theorem 9. Then there exists a unique global in time solution $\overline{U}_\epsilon = (\bar\rho_\epsilon, \bar u_\epsilon)$ of the approximate system (143)-(145) deduced from a solution of the KZK equation with the help of (104), (105), (139). The function $\overline{U}_\epsilon(x_1,x',t) = \overline{U}_\epsilon(x_1-ct, \epsilon x_1, \sqrt{\epsilon}x')$, given by the formula (110), is defined in the half space*

$$\{x_1>0,\ x'\in\mathcal{R}^{n-1},\ t>0\}.$$

Moreover, according to its definition, we have (148), (149).

Then there exists a constant C such that for all rather small ϵ the solutions $(\rho_\epsilon, u_\epsilon)$ of (136) and $(\overline{\rho}_\epsilon, \overline{u}_\epsilon)$ of (143)-(145) satisfy the following stability estimate

$$\|\rho_\epsilon-\overline{\rho}_\epsilon\|_{L_2}+\|\rho_\epsilon u_\epsilon-\overline{\rho}_\epsilon\overline{u}_\epsilon\|_{L_2}\leq \epsilon^{\frac{5}{2}}e^{C\|\nabla.\overline{U}_\epsilon\|_{L_\infty}t}\leq \epsilon^{\frac{5}{2}}e^{C\epsilon t}. \tag{152}$$

which remains any finite time

$$0<t<\frac{T}{\epsilon}\ln\frac{1}{\epsilon}$$

smaller than the order ϵ (here T is a positive constant and $T=O(1)$).

Proof. The approximation result.
So we have two systems

$$\begin{cases}\partial_t\eta(\widetilde{U}_\epsilon)+\nabla.q(\widetilde{U}_\epsilon)-\epsilon\nu\begin{bmatrix}0\\ u_\epsilon\triangle u_\epsilon\end{bmatrix}=0,\\ \partial_t\widetilde{U}_\epsilon+\nabla.F(\widetilde{U}_\epsilon)-\epsilon\nu\begin{bmatrix}0\\ \triangle u_\epsilon\end{bmatrix}=0,\end{cases}$$

$$\begin{cases}\partial_t\eta(\overline{U}_\epsilon)+\nabla.q(\overline{U}_\epsilon)-\epsilon\nu\begin{bmatrix}0\\ \bar u_\epsilon\triangle \bar u_\epsilon\end{bmatrix}=\epsilon^{\frac{5}{2}}\left(\frac{\eta(\overline{U}_\epsilon)+p(\bar\rho_\epsilon)}{\bar\rho_\epsilon}R_1+\bar u_\epsilon\vec{R}_2\right),\\ \partial_t\overline{U}_\epsilon+\nabla.F(\overline{U}_\epsilon)-\epsilon\nu\begin{bmatrix}0\\ \triangle \bar u_\epsilon\end{bmatrix}=\epsilon^{\frac{5}{2}}\vec{R},\end{cases}$$

where $\vec{R}=(R_1,\vec{R_2})$ is the rest from (146). Since we suppose that $\overline{U}_\epsilon = (\bar\rho_\epsilon, \bar\rho_\epsilon\bar u_\epsilon)^T$ is bounded we can denote again

$$\bar R=\frac{\eta(\overline{U}_\epsilon)+p(\bar\rho_\epsilon)}{\bar\rho_\epsilon}R_1+\bar u_\epsilon\vec{R}_2.$$

Let us compute

$$\frac{\partial}{\partial t}(\eta(\widetilde{U}_\epsilon)-\eta(\overline{U}_\epsilon)-\eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon-\overline{U}_\epsilon)).$$

The first we find from initial systems that

$$\frac{\partial}{\partial t}(\eta(\widetilde{U}_\epsilon)-\eta(\overline{U}_\epsilon)) = -\nabla.((q(\widetilde{U}_\epsilon)-q(\overline{U}_\epsilon))+\epsilon\nu\begin{bmatrix} 0 \\ u_\epsilon\triangle u_\epsilon-\bar{u}_\epsilon\triangle\bar{u}_\epsilon\end{bmatrix}-\epsilon^{\frac{5}{2}}\bar{R}. \quad (153)$$

Then we notice that

$$-\frac{\partial}{\partial t}\left(\eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon-\overline{U}_\epsilon)\right) = -\left(\frac{\partial\overline{U}_\epsilon}{\partial t}\right)^T\eta''(\overline{U}_\epsilon)(\widetilde{U}_\epsilon-\overline{U}_\epsilon)-\eta'(\overline{U}_\epsilon)\left(\frac{\partial\widetilde{U}_\epsilon}{\partial t}-\frac{\partial\overline{U}_\epsilon}{\partial t}\right),$$

where

$$-\left(\frac{\partial\overline{U}_\epsilon}{\partial t}\right)^T\eta''(\overline{U}_\epsilon)(\widetilde{U}_\epsilon-\overline{U}_\epsilon) = -\left(-\nabla.F(\overline{U}_\epsilon)+\begin{bmatrix} 0 \\ \epsilon\nu\triangle\bar{u}_\epsilon\end{bmatrix}+\epsilon^{\frac{5}{2}}\vec{R}\right)^T\eta''(\overline{U}_\epsilon)\left(\widetilde{U}_\epsilon-\overline{U}_\epsilon\right),$$

and

$$\begin{aligned}-\eta'(\overline{U}_\epsilon)\left(\frac{\partial\widetilde{U}_\epsilon}{\partial t}-\frac{\partial\overline{U}_\epsilon}{\partial t}\right) &= -\eta'(\overline{U}_\epsilon)\left(-\nabla.F(\widetilde{U}_\epsilon)+\nabla.F(\overline{U}_\epsilon)\right)- \\ &\quad -\eta'(\overline{U}_\epsilon)\epsilon\nu\begin{bmatrix} 0 \\ \triangle u_\epsilon-\triangle\bar{u}_\epsilon\end{bmatrix}+\epsilon^{\frac{5}{2}}\eta'(\overline{U}_\epsilon)\vec{R}.\end{aligned}$$

Using the property for convex entropy $\eta''(U)F'(U) = (F'(U))^T\eta''(U)$ we find that

$$\left(\nabla.F(\overline{U}_\epsilon)\right)^T\eta''(\overline{U}_\epsilon)(\widetilde{U}_\epsilon-\overline{U}_\epsilon) = \nabla.\overline{U}_\epsilon^T(F'(\overline{U}_\epsilon))^T\eta''(\overline{U}_\epsilon)(\widetilde{U}_\epsilon-\overline{U}_\epsilon) = \nabla.\overline{U}_\epsilon^T\eta''(\overline{U}_\epsilon)F'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon-\overline{U}_\epsilon).$$

So we obtain that

$$\begin{aligned}&\frac{\partial}{\partial t}(\eta(\widetilde{U}_\epsilon)-\eta(\overline{U}_\epsilon)-\eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon-\overline{U}_\epsilon)) = -\nabla.(q(\widetilde{U}_\epsilon)-q(\overline{U}_\epsilon))+\epsilon\nu\begin{bmatrix} 0 \\ u_\epsilon\triangle u_\epsilon-\bar{u}_\epsilon\triangle\bar{u}_\epsilon\end{bmatrix}-\epsilon^{\frac{5}{2}}\bar{R}+ \\ &+\nabla.\overline{U}_\epsilon^T\eta''(\overline{U}_\epsilon)F'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon-\overline{U}_\epsilon)-\begin{bmatrix} 0 \\ \epsilon\nu\triangle\bar{u}_\epsilon\end{bmatrix}^T\eta''(\overline{U}_\epsilon)\left(\widetilde{U}_\epsilon-\overline{U}_\epsilon\right)-\epsilon^{\frac{5}{2}}\vec{R}^T\eta''(\overline{U}_\epsilon)\left(\widetilde{U}_\epsilon-\overline{U}_\epsilon\right)+ \\ &-\eta'(\overline{U}_\epsilon)\left(-\nabla.F(\widetilde{U}_\epsilon)+\nabla.F(\overline{U}_\epsilon)\right)-\eta'(\overline{U}_\epsilon)\epsilon\nu\begin{bmatrix} 0 \\ \triangle u_\epsilon-\triangle\bar{u}_\epsilon\end{bmatrix}+\epsilon^{\frac{5}{2}}\eta'(\overline{U}_\epsilon)\vec{R}, \qquad (154)\end{aligned}$$

where, thanks to (150), (151) (and $\frac{\rho}{\bar{\rho}_\epsilon} = s+1$),

$$-\begin{bmatrix} 0 \\ \epsilon\nu\triangle\bar{u}_\epsilon\end{bmatrix}^T\eta''(\overline{U}_\epsilon)\left(\widetilde{U}_\epsilon-\overline{U}_\epsilon\right) = -\epsilon\nu\begin{bmatrix} -\frac{\bar{u}_\epsilon}{\bar{\rho}_\epsilon}\triangle\bar{u}_\epsilon \\ \frac{1}{\bar{\rho}_\epsilon}\triangle\bar{u}_\epsilon\end{bmatrix}^T\begin{bmatrix} \rho_\epsilon-\bar{\rho}_\epsilon \\ \rho u-\bar{\rho}_\epsilon\bar{u}_\epsilon\end{bmatrix} =$$

$$= -\epsilon\nu\triangle\bar{u}_\epsilon\,(-\bar{u}_\epsilon,1)\begin{bmatrix} s \\ u(s+1)-\bar{u}_\epsilon\end{bmatrix} = -\epsilon\nu\triangle\bar{u}_\epsilon(u_\epsilon-\bar{u}_\epsilon)-\epsilon\nu\triangle\bar{u}_\epsilon\frac{\rho_\epsilon-\bar{\rho}_\epsilon}{\bar{\rho}_\epsilon}(u_\epsilon-\bar{u}_\epsilon),$$

$$-\eta'(\overline{U}_\epsilon)\epsilon\nu\left[\begin{array}{c}0\\ \triangle u_\epsilon-\triangle\bar{u}_\epsilon\end{array}\right]=-\epsilon\nu\bar{u}_\epsilon(\triangle u_\epsilon-\triangle\bar{u}_\epsilon)$$

Integrate (117) over the half space. The use of the integration by parts gives with notation q_1, F_1 for first components of vectors q and F:

$$\begin{aligned}
&\frac{d}{dt}\int_{x_1>0}(\eta(\widetilde{U}_\epsilon)-\eta(\overline{U}_\epsilon)-\eta'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon-\overline{U}_\epsilon)dx=\\
&-\int_{x_1=0}(q_1(\widetilde{U}_\epsilon)-q_1(\overline{U}_\epsilon)-\eta'(\overline{U}_\epsilon)^T(F_1(\widetilde{U}_\epsilon)-F_1(\overline{U}_\epsilon)))dx'-\\
&-\int_{x_1>0}\nabla.\overline{U}_\epsilon^T\eta''(\overline{U}_\epsilon)(F(\widetilde{U}_\epsilon)-F(\overline{U}_\epsilon)-F'(\overline{U}_\epsilon)(\widetilde{U}_\epsilon-\overline{U}_\epsilon))dx+\\
&+\epsilon\nu\int_{x_1=0}\left(u_\epsilon\frac{\partial u_\epsilon}{\partial x_1}-\bar{u}_\epsilon\frac{\partial\bar{u}_\epsilon}{\partial x_1}\right)dx'-\epsilon\nu\int_{x_1>0}\left(|\nabla.u_\epsilon|^2-|\nabla.\bar{u}_\epsilon|^2\right)dx+\\
&-\epsilon^{\frac{5}{2}}\int_{x_1>0}\left(\bar{R}-\eta'(\overline{U}_\epsilon)\vec{R}\right)dx-\epsilon^{\frac{5}{2}}\int_{x_1>0}\vec{R}^T\eta''(\overline{U}_\epsilon)\left(\widetilde{U}_\epsilon-\overline{U}_\epsilon\right)dx+\\
&+\epsilon\nu\int_{x_1>0}\triangle\bar{u}_\epsilon\frac{\rho_\epsilon-\bar{\rho}_\epsilon}{\bar{\rho}_\epsilon}(u_\epsilon-\bar{u}_\epsilon)dx+\epsilon\nu\int_{x_1=0}\left(-\bar{u}_\epsilon\frac{\partial(u_\epsilon-\bar{u}_\epsilon)}{\partial x_1}-\frac{\partial\bar{u}_\epsilon}{\partial x_1}(u_\epsilon-\bar{u}_\epsilon)\right)dx'+\\
&+\epsilon\nu\int_{x_1>0}\left(\nabla.\bar{u}_\epsilon\cdot\nabla.(u_\epsilon-\bar{u}_\epsilon)+\nabla.\bar{u}_\epsilon\cdot\nabla.(u_\epsilon-\bar{u}_\epsilon)\right)dx.
\end{aligned}\tag{155}$$

It is easy to see that

$$\begin{aligned}
&\epsilon\nu\int_{x_1=0}\left(u_\epsilon\frac{\partial u_\epsilon}{\partial x_1}-\bar{u}_\epsilon\frac{\partial\bar{u}_\epsilon}{\partial x_1}\right)dx'+\epsilon\nu\int_{x_1=0}\left(-\bar{u}_\epsilon\frac{\partial(u_\epsilon-\bar{u}_\epsilon)}{\partial x_1}-\frac{\partial\bar{u}_\epsilon}{\partial x_1}(u_\epsilon-\bar{u}_\epsilon)\right)dx'=\\
&=\epsilon\nu\int_{x_1=0}(u_\epsilon-\bar{u}_\epsilon)\frac{\partial(u_\epsilon-\bar{u}_\epsilon)}{\partial x_1}dx'
\end{aligned}\tag{156}$$

and

$$-\epsilon\nu\int_{x_1>0}\left(|\nabla.u_\epsilon|^2-|\nabla.\bar{u}_\epsilon|^2\right)dx+\epsilon\nu\int_{x_1>0}\left(\nabla.\bar{u}_\epsilon\cdot\nabla.(u_\epsilon-\bar{u}_\epsilon)+\nabla.\bar{u}_\epsilon\cdot\nabla.(u_\epsilon-\bar{u}_\epsilon)\right)dx=$$

$$=-\epsilon\nu\int_{x_1>0}|\nabla.(u_\epsilon-\bar{u}_\epsilon)|^2dx.$$

Let us envisage now the boundary condition

$$\begin{aligned}
&-\int_{x_1=0}(q_1(\widetilde{U}_\epsilon)-q_1(\overline{U}_\epsilon)-\eta'(\overline{U}_\epsilon)^T(F_1(\widetilde{U}_\epsilon)-F_1(\overline{U}_\epsilon)))dx'=\\
&=(q_1(\widetilde{U}_\epsilon)-q_1(\overline{U}_\epsilon)-\eta'(\overline{U}_\epsilon)^T(F_1(\widetilde{U}_\epsilon)-F_1(\overline{U}_\epsilon)))|_{x_1=0}.
\end{aligned}\tag{157}$$

Explicitly (157) has the form:

$$(q_1(\widetilde{U}_\epsilon)-q_1(\overline{U}_\epsilon)-\eta'(\overline{U}_\epsilon)^T(F_1(\widetilde{U}_\epsilon)-F_1(\overline{U}_\epsilon)))|_{x_1=0}=u_1(\eta(\widetilde{U}_\epsilon)+p(\rho_\epsilon))-\bar{u}_1(\eta(\overline{U}_\epsilon)+p(\bar{\rho}_\epsilon))-$$

$$- \begin{bmatrix} H'(\bar\rho_\epsilon) - \frac{\bar u_\epsilon^2}{2} \\ \bar u_1 \end{bmatrix}^T \begin{bmatrix} \rho_\epsilon u_1 - \bar\rho_\epsilon \bar u_1 \\ \rho_\epsilon u_\epsilon^2 + p(\rho_\epsilon) - \bar\rho_\epsilon \bar u_\epsilon^2 - p(\bar\rho_\epsilon) \end{bmatrix}\Bigg|_{x_1=0}.$$

Choosing always $u_1|_{x_1=0} = \bar u_1|_{x_1=0}$ we obtain with the help of the facts that the entropy η is convex

$$\eta(\widetilde U_\epsilon) - \eta(\overline U_\epsilon) - \eta'(\overline U_\epsilon)(\widetilde U_\epsilon - \overline U_\epsilon) \geq \alpha|\widetilde U_\epsilon - \overline U_\epsilon|^2,$$

$$(q_1(\widetilde U_\epsilon) - q_1(\overline U_\epsilon) - \eta'(\overline U_\epsilon)^T (F_1(\widetilde U_\epsilon) - F_1(\overline U_\epsilon)))|_{x_1=0} = u_1(\eta(\widetilde U_\epsilon) - \eta(\overline U_\epsilon) + p(\rho_\epsilon) - p(\bar\rho_\epsilon)) -$$
$$-u_1(\rho_\epsilon - \bar\rho_\epsilon)\left(H'(\bar\rho_\epsilon) - \frac{\bar u_\epsilon^2}{2}\right) - u_1\left[\rho_\epsilon u_\epsilon^2 - \bar\rho_\epsilon \bar u_\epsilon^2 + p(\rho_\epsilon) - p(\bar\rho_\epsilon)\right]|_{x_1=0} =$$
$$= u_1\left(\eta(\widetilde U_\epsilon) - \eta(\overline U_\epsilon) - \eta'(\overline U_\epsilon)^T(\widetilde U_\epsilon - \overline U_\epsilon)\right)\Big|_{x_1=0} \tag{158}$$

and so this boundary condition take the same sign as the first component of velocity $u_1|_{x_1=0}$ since

$$\eta(\widetilde U_\epsilon) - \eta(\overline U_\epsilon) - \eta'(\overline U_\epsilon)^T(\widetilde U_\epsilon - \overline U_\epsilon) \geq (\widetilde U_\epsilon - \overline U_\epsilon)^2 \geq 0$$

is always positive.

So for the boundary conditions we take $u_\epsilon|_{x_1=0} = \bar u_\epsilon|_{x_1=0}$ and if $u_1 > 0$ we also suppose $\rho_\epsilon|_{x_1=0} = \bar\rho_\epsilon|_{x_1=0}$. Then the first boundary condition (156) is always zero:

$$\epsilon\nu\int_{x_1=0}(u_\epsilon - \bar u_\epsilon)\frac{\partial(u_\epsilon - \bar u_\epsilon)}{\partial x_1}dx' = -\epsilon\nu(u_\epsilon - \bar u_\epsilon)\frac{\partial(u_\epsilon - \bar u_\epsilon)}{\partial x_1}|_{x_1=0} = 0.$$

For the second boundary condition (158), if $u_\epsilon|_{x_1=0} < 0$ then we have in the left-hand side of the estimate a positive quantity on the boundary and it can be omitted.

If $u_\epsilon|_{x_1=0} > 0$ then we have for $\rho_\epsilon|_{x_1=0} = \bar\rho_\epsilon|_{x_1=0}$

$$u_1\left(H(\rho_\epsilon) + \rho_\epsilon\frac{u_\epsilon^2}{2} - H(\bar\rho_\epsilon) - \bar\rho_\epsilon\frac{\bar u_\epsilon^2}{2} - (\rho_\epsilon - \bar\rho_\epsilon)\left(H'(\bar\rho_\epsilon) - \frac{\bar u_\epsilon^2}{2}\right) - \rho_\epsilon u_\epsilon^2 + \bar\rho_\epsilon \bar u_\epsilon^2\right)\Bigg|_{x_1=0} =$$

$$= u_1\left(H(\rho_\epsilon) - H(\bar\rho_\epsilon) - (\rho_\epsilon - \bar\rho_\epsilon)H'(\bar\rho_\epsilon) + \rho_\epsilon\left(\frac{\bar u_\epsilon^2}{2} - \frac{u_\epsilon^2}{2}\right)\right)\Bigg|_{x_1=0} =$$
$$= u_1\left(H(\rho_\epsilon) - H(\bar\rho_\epsilon) - (\rho_\epsilon - \bar\rho_\epsilon)H'(\bar\rho_\epsilon)\right)|_{x_1=0} = 0.$$

We use the Taylor expansion

$$F(\widetilde U_\epsilon) - F(\overline U_\epsilon) = F'(\overline U_\epsilon)(\widetilde U_\epsilon - \overline U_\epsilon) + O(|\widetilde U_\epsilon - \overline U_\epsilon|^2),$$

and we have

$$F(\widetilde U_\epsilon) - F(\overline U_\epsilon) - F'(\overline U_\epsilon)(\widetilde U_\epsilon - \overline U_\epsilon) \leq C|\widetilde U_\epsilon - \overline U_\epsilon|^2.$$

Taking the same initial data

$$\overline{U}_\epsilon|_{t=0} = \widetilde{U}_\epsilon|_{t=0},$$

we obtain

$$\frac{d}{dt}\int_{x_1>0} |\widetilde{U}_\epsilon - \overline{U}_\epsilon|^2 dx \le C\|\nabla.\overline{U}_\epsilon\|_{L_\infty}\int_{x_1>0} |\widetilde{U}_\epsilon - \overline{U}_\epsilon|^2 dx + K\epsilon^5.$$

Here the constants K and C do not depend on t.

Therefore applying the Gronwall lemma since $\nabla.\overline{U}_\epsilon = O(\epsilon)$ one has

$$\|\widetilde{U}_\epsilon - \overline{U}_\epsilon\|^2_{L_2(\{x_1>0\})} \le K\epsilon^5 e^{\epsilon Ct}. \tag{159}$$

As soon as the difference of the solutions has the order $\widetilde{U}_\epsilon - \overline{U}_\epsilon = O(\epsilon)$ and the left side of (159) has the order $O(\epsilon^2)$, so we would like to have the inequality

$$\epsilon^5 \int_0^t e^{C\epsilon(t-s)} ds < \epsilon^2.$$

From $\epsilon^3 e^{C\epsilon t} < 1$ we obtain that the estimate (152) is smaller than the order ϵ^2 for time $t < T\frac{1}{\epsilon}\ln\frac{1}{\epsilon}$. $\square$

It is possible to extend the previous results to the case where $\widetilde{U}_\epsilon$ is only an admissible solution satisfying the boundary conditions.

Definition 1 *The pair of functions (ρ, u) is called an admissible weak solution of Navier-Stokes system (136) satisfying the boundary conditions in the half space if it satisfies the following properties:*

1. *it is a weak solution of (136),*
2. *it satisfies in the sense of distributions (see [14, p.52])*

$$\partial_t\eta(U_\epsilon) + \nabla.q(U_\epsilon) - \epsilon\nu\begin{bmatrix} 0 \\ u_\epsilon \triangle u_\epsilon \end{bmatrix} \le 0,$$

or equivalently for all nonnegative twice differentiable test function ψ with compact support in the half space

$$\int_0^T\int_{x_1>0}\left(\partial_t\psi\eta(U_\epsilon) + \nabla.\psi q(U_\epsilon) - \epsilon\nu\begin{bmatrix} 0 \\ |\nabla.u_\epsilon|^2 \end{bmatrix}\psi + \epsilon\nu\begin{bmatrix} 0 \\ \nabla\frac{u_\epsilon^2}{2} \end{bmatrix}\nabla.\psi\right)dxdt +$$
$$+\int_{x_1>0}\psi(x,0)\eta(U_0(x))dx + \int_0^T\int_{\mathcal{R}^{n-1}}\psi\left(q(U_\epsilon) - \epsilon\nu\begin{bmatrix} 0 \\ u_\epsilon\cdot\nabla.u_\epsilon \end{bmatrix}\right)\Bigg|_{x_1=0} dx'dt \ge 0.$$

3. *it satisfies the equality*

$$-\int_{x_1>0}\frac{U_\epsilon^2}{2}dx + \int_0^t\int_{x_1>0}\left(\nabla.U_\epsilon F(U_\epsilon) + \epsilon\nu\begin{bmatrix} 0 \\ |\nabla.u_\epsilon|^2 \end{bmatrix}\right)dxds + \int_{x_1>0}U_0^2(x)dx +$$
$$+\int_0^t\int_{\mathcal{R}^{n-1}}U_\epsilon\left(F(U_\epsilon) - \epsilon\nu\begin{bmatrix} 0 \\ \nabla u_\epsilon \end{bmatrix}\right)\Bigg|_{x_1=0} dx'dt = 0.$$

Theorem 11 *To have the estimate (152) it is sufficient to have an admissible weak solution of the Navier-Stokes system (136) satisfying the boundary conditions in the half space*

$$\partial_t U_\epsilon + \nabla . F(U_\epsilon) - \epsilon\nu \begin{bmatrix} 0 \\ \triangle u_\epsilon \end{bmatrix} = 0,$$

such that [28, 31] $\rho_\epsilon \in L_\infty((0,T), L_2) \cap C([0,T], L_p)$ *for* $1 \leq p \leq 2$, $\rho_\epsilon \geq 0$ *a.e.,* $\nabla . u_\epsilon \in L_2((0,T); L_2)$, $\rho_\epsilon |u_\epsilon|^2 \in L_\infty((0,T), L_1)$, $u_\epsilon \in L_2((0,T), H_0^1)$, $\rho_\epsilon u_\epsilon \in C([0,T], L_{\frac{4}{3}} - \omega)$, *here by* $C([0,T], L_{\frac{4}{3}} - \omega)$ *is denoted the space of continuous functions with values in a closed ball of* $L_{\frac{4}{3}}$ *endowed with the weak topology.*

3.3 Conclusion

The approximation result for nonlinear KZK equation

$$\|\bar{U}_\epsilon - U_\epsilon\|^2_{L_2} \leq \epsilon^5 e^{C\epsilon t}$$

is valid in the viscous and non viscous cases. The obtained estimate guarantees that the difference $\bar{U}_\epsilon - U_\epsilon$ stays of order $O(\epsilon)$ during the time of the order $\frac{1}{\epsilon} \ln \frac{1}{\epsilon}$.

Let us for the conclusion make a comparative table for the approximation results (see table 1).

One use the notation: $A(u - v) \equiv |\partial_t (u - v)|^2 + |\nabla . (u - v)|^2$.

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Table 1: The approximation's results

	Linearization of Navier-Stokes system	Navier-Stokes system	Euler system
Domain	the half space $\{x_1 > 0, x' \in \mathcal{R}^{n-1}\}$		the cone $Q_\epsilon(t) = \{\lvert x_1\rvert < \frac{R}{\epsilon} - ct\} \times \mathcal{R}^{n-1}_{x'}$
Exact system	$\partial_t \rho + \nabla . u = 0,$ $\partial_t u + \nabla p(\rho) = \epsilon\nu\Delta u$	$\partial_t \rho_\epsilon + \mathrm{div}(\rho_\epsilon u_\epsilon) = 0,$ $\rho_\epsilon[\partial_t u_\epsilon + (u_\epsilon \cdot \nabla)u_\epsilon] = -\nabla p(\rho_\epsilon) + \epsilon\nu\Delta u_\epsilon$	$\partial_t \rho_\epsilon + \mathrm{div}(\rho_\epsilon u_\epsilon) = 0,$ $\rho_\epsilon[\partial_t u_\epsilon + (u_\epsilon \cdot \nabla)u_\epsilon] + \nabla p(\rho_\epsilon) = 0$
State equation	$p(\rho) = c^2\rho$	$p = p(\rho_\epsilon) = c^2\epsilon\rho_\epsilon + \frac{(\gamma-1)c^2}{2\rho_0}\epsilon^2{\rho_\epsilon}^2$	
Exact solution	$U_\epsilon = (\rho, u)$	$U_\epsilon = (\rho_\epsilon, \rho_\epsilon u_\epsilon)$	
Approximate solution	$\bar{U}_\epsilon = (I, v + \sqrt{\epsilon}\vec{w})$	$\bar{U}_\epsilon = (\rho_0 + \epsilon I, \epsilon(\rho_0 + \epsilon I)(v + \epsilon v_1, \sqrt{\epsilon}\vec{w}))$	
$v_1(\tau, z, y) =$	0	$-\frac{c^2}{\rho_0}\Big(\int_0^\tau \partial_z I ds + \int_0^L \frac{s}{L}\partial_z I ds\Big) +$ $+ \frac{(\gamma-1)}{2\rho_0^2}cI^2 - \frac{c(\gamma-1)}{2L\rho_0^2}\int_0^L I^2 d\tau$	$-\frac{c^2}{\rho_0}\Big(\int_0^\tau \partial_z I ds + \int_0^L \frac{s}{L}\partial_z I ds\Big) +$ $+ \frac{(\gamma-1)}{2\rho_0^2}cI^2 - \frac{c(\gamma-1)}{2L\rho_0^2}\int_0^L I^2 d\tau + \frac{\nu}{c\rho_0^2}\partial_\tau I$
I is solution of	$2c\partial^2_{\tau z} I - c^2\triangle_y I -$ $-\frac{\nu}{\rho_0 c^2}\partial^3_\tau I = 0$	$c\partial^2_{\tau z} I - \frac{(\gamma+1)}{4\rho_0}\partial^2_\tau I^2$- $-\frac{\nu}{2c^2\rho_0}\partial^3_\tau I - \frac{c^2}{2}\Delta_y I = 0$	$c\partial^2_{\tau z} I - \frac{(\gamma+1)}{4\rho_0}\partial^2_\tau I^2 - \frac{c^2}{2}\Delta_y I = 0$
Smoothness of KZK solution	$s > \max\{6, [\frac{n}{2}] + 1\}$		$s > \max\{4, [\frac{n}{2}] + 1\}$
Time of validation for $U_\epsilon - \overline{U}_\epsilon = O(\epsilon)$	$T\frac{1}{\epsilon^2}$ for $\bar{u}_\epsilon - u_\epsilon$, $T \ln\frac{1}{\epsilon}$ for $\bar{\rho}_\epsilon - \rho_\epsilon$	$\frac{T}{\epsilon}\ln\frac{1}{\epsilon}$	
Estimation	$\int_{x_1>0} A(\bar{u}_\epsilon - u_\epsilon)dx \le \epsilon^4 t^2$ $\lVert\bar{\rho}_\epsilon - \rho_\epsilon\rVert_{L_2} \le \epsilon^2 e^{Ct}$	$\lVert\bar{U}_\epsilon - U_\epsilon\rVert^2_{L_2} \le \epsilon^5 e^{C\epsilon t}$	$s' = s - 4 \quad \lVert\bar{U}_\epsilon - U_\epsilon\rVert^2_{H^{s'}} \le \epsilon^5 e^{C\epsilon t}$

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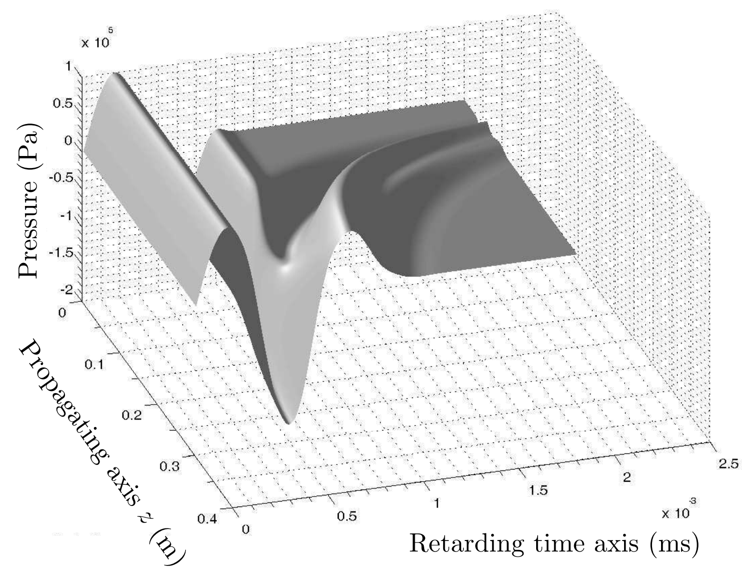

Figure 2: The effects of diffraction (33) in $3d$ for the pressure with a square source $3\,\mathrm{cm} \times 3\,\mathrm{cm}$. The parameters of simulation for propagation in water: $\triangle y_1 = \triangle y_2 = 3.75 \times 10^{-4}\,\mathrm{m}$, $\triangle z = 6 \times 10^{-4}\,\mathrm{m}$, $\triangle \tau = 6.6667 \times 10^{-9}\,\mathrm{s}$, $\rho_0 = 1000\,\mathrm{kg/m^3}$, $c = 1500\,\mathrm{m/s}$.

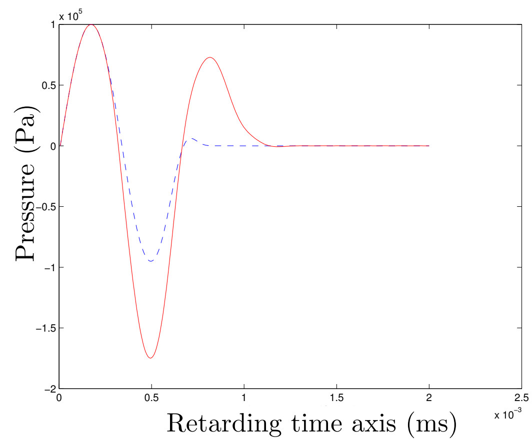

Figure 3: The effects of diffraction (33) along the retarded axis τ in the two different places of the propagation axis z with a square source $3\,\mathrm{cm} \times 3\,\mathrm{cm}$. The dotted line corresponds to the signal source $z = 0$ and the solid line to the pressure at the distance $z = 30\,\mathrm{cm}$. The parameters of simulation are the same as in figure 2.

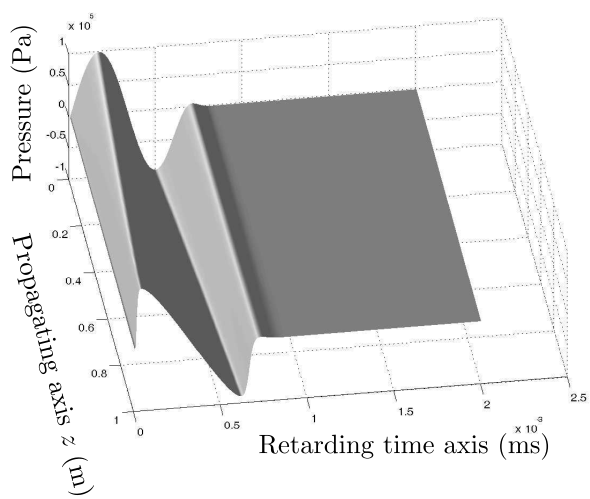

Figure 4: The effects of absorption and nonlinear effects (36) in $3d$ for the pressure with a square source $3\,\mathrm{cm} \times 3\,\mathrm{cm}$. The parameters of simulation for propagation in water: $\triangle y_1 = \triangle y_2 = 3.75 \times 10^{-4}\,\mathrm{m}$, $\triangle z = 5 \times 10^{-3}\,\mathrm{m}$, $\triangle \tau = 6.6667 \times 10^{-9}\,\mathrm{s}$, $\beta = 5$, $\delta = 4.1 \times 10^{-6}\,\mathrm{Np/m}$, $\rho_0 = 1000\,\mathrm{kg/m^3}$, $c = 1500\,\mathrm{m/s}$.

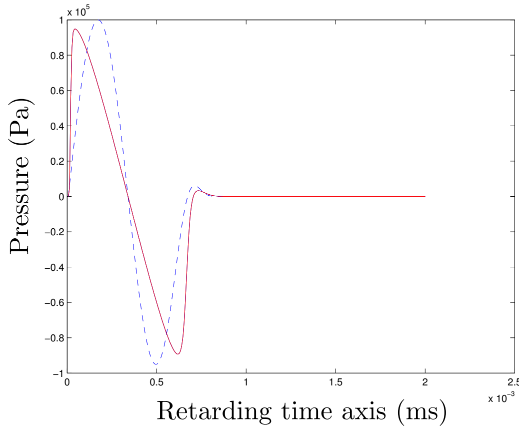

Figure 5: The effects of absorption and nonlinear effects (36) along the retarded axis τ in the two different places of the propagation axis z with a square source $3\,\text{cm} \times 3\,\text{cm}$. The dotted line corresponds to the signal source $z = 0$ and the solid line to the pressure at the distance $z = 1\,\text{m}$. The parameters of simulation are the same as in figure 4.

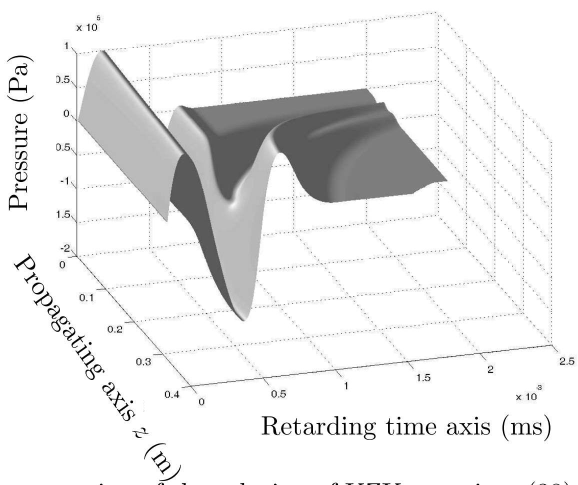

Figure 6: Representation of the solution of KZK equation (38) for the pressure in $3d$ with a square source $3\,\text{cm} \times 3\,\text{cm}$. The parameters of simulation for propagation in water: $\triangle y_1 = \triangle y_2 = 3.75 \times 10^{-4}\,\text{m}$, $\triangle z = 1 \times 10^{-3}\,\text{m}$, $\triangle\tau = 6.6667 \times 10^{-9}\,\text{s}$, $\beta = 5$, $\delta = 4.1 \times 10^{-6}\,\text{Np/m}$, $\rho_0 = 1000\,\text{kg/m}^3$, $c = 1500\,\text{m/s}$.

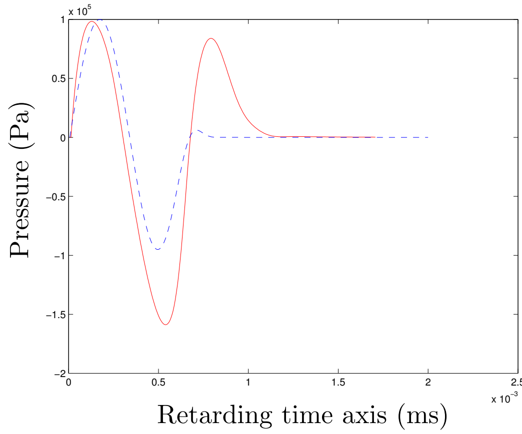

Figure 7: Representation of the solution of KZK equation (38) along the retarded axis τ in the two different places of the propagation axis z with a square source $3\,\mathrm{cm} \times 3\,\mathrm{cm}$. The dotted line corresponds to the signal source $z = 0$ and the solid line to the pressure at the distance $z = 31.2\,\mathrm{cm}$. The parameters of simulation are the same as in figure 6.

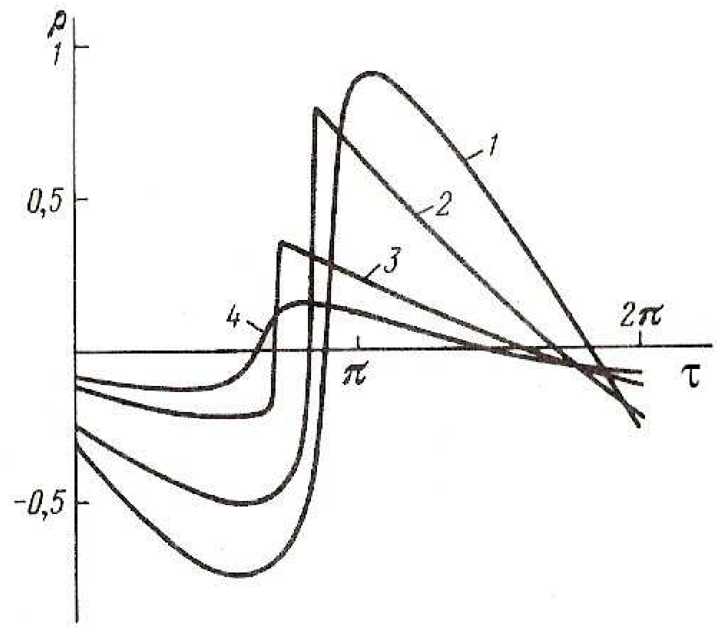

Figure 8: Profiles of the solution of KZK equation (39) for the density along the axis τ with different values of z. The values of z on the curves 1-4 respectively are 0.15, 0.3, 0.7, 1.2; $N = 3.25$, $\delta = 0.1$ (see [10, p.80]).

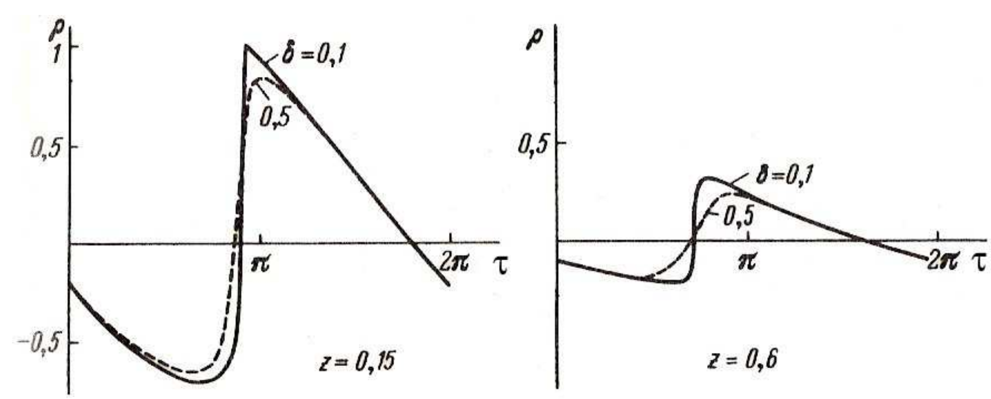

Figure 9: Wave profiles of the solution of KZK equation (39) corresponding to different δ; $N = 5$ (see [10, p.81]).